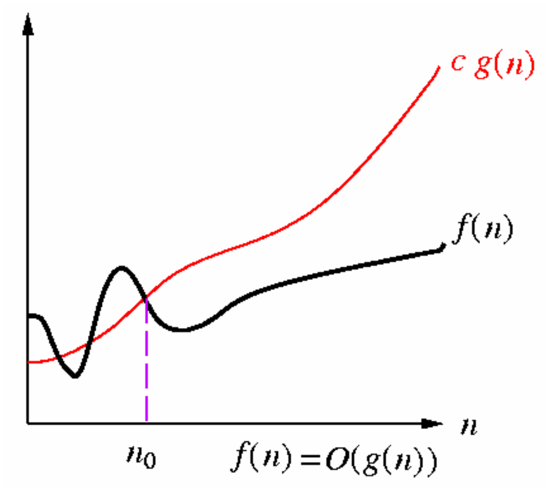
Unit 3: Computational Complexity

- Course contents:
  - Computational complexity
  - NP-completeness
  - General Purpose Combinational Optimizations
- Readings
  - Chapters 3, 4, and 5

Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

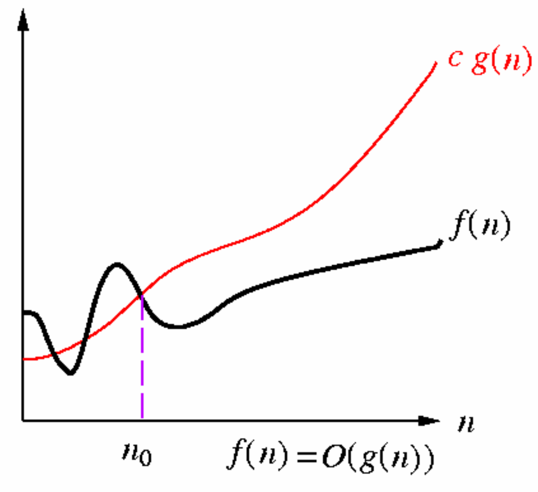
O: Upper Bounding Function

- **Def:** $f(n) = O(g(n))$ if $\exists\, c > 0$ and $n_0 > 0$ such that $0 \le$ ***f*(*n*)** $\le$ ***cg*(*n*)** for all $n \ge n_0$.
  - Examples: $2n^2 + 3n = O(n^2)$, $2n^2 = O(n^3)$, $3n \lg n = O(n^2)$
- Intuition: $f(n)$ "$\le$ " $g(n)$ when we ignore constant multiples and small values of n.



Big-*O* Notation

- How to show *O* (Big-Oh) relationships?
  - $f(n) = O(g(n))$ iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some $c \geq 0$.
- "An algorithm has worst-case running time $O(f(n))$": there is a constant *c* s.t. for every *n* big enough, **every execution** on an input of size *n* takes **at most** $cf(n)$ time.



Big-Theta Notation

- **Def:** $f(n) = \Theta(g(n))$ if $\exists\ c_1 > 0\ c_2 > 0$ and $n_0 > 0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.
  - Examples: $2n^2 + 3n = \Theta(n^2)$, $2n^2 = O(n^2)$, $3n \lg n = O(n \lg n)$
  - g(n) is asymptotically tight bound of f(n)

Computational Complexity

- Computational complexity: an abstract measure of the time and space necessary to execute an algorithm as functions of its "input size".
- Input size examples:
  - sort n words of bounded length $\Rightarrow n$
  - **the input is the integer $n \Rightarrow \lg n$**
  - the input is the graph $G(V, E) \Rightarrow |V|$ and $|E|$
- Time complexity is expressed in *elementary computational steps* (e.g., an addition, multiplication, pointer indirection).
- Space Complexity is expressed in *memory locations* (e.g. bits, bytes, words).

Asymptotic Functions

- Polynomial-time complexity: $O(n^k)$, where n is the **input size** and k is a constant (k = O(1)).
- Example polynomial functions:
  - 999: constant
  - $\lg n$: logarithmic
  - $\sqrt{n}$: sublinear
  - n: linear
  - $n \lg n$: loglinear
  - n^2: quadratic
  - n^3: cubic
- Example non-polynomial functions
  - 2^n, 3^n: exponential
  - $n!$: factorial

Running-time Comparison

- Assume 1000 MIPS (Yr: 200x), 1 instruction /operation

Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

Ch4. Tractable and Intractable Problems

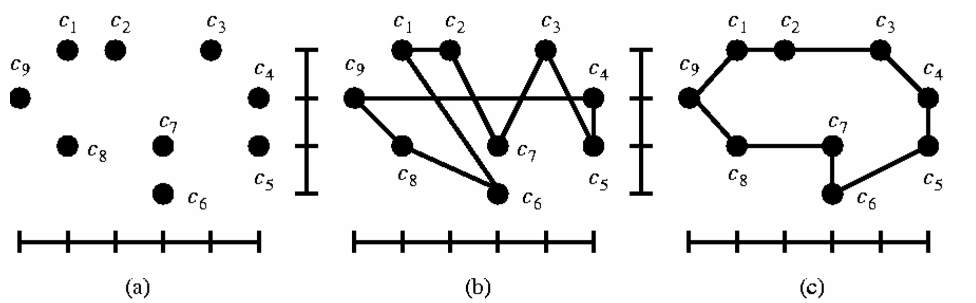
- Tractable problems
  - Can be solved within polynomial time
- Intractable problems
  - Cannot be solved within polynomial time
- NP-complete problems
  - Likely to be intractable
  - Still under research ...

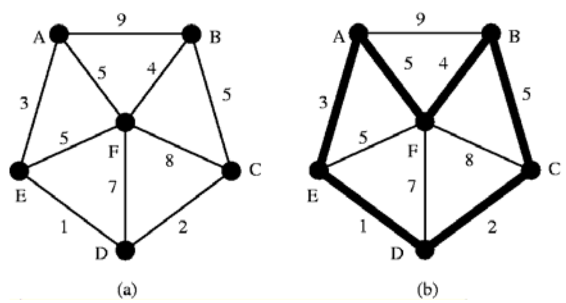
Optimization Problems

- **Problem:** a general class, e.g., "the shortest-path problem for directed acyclic graphs."
- **Instance:** a specific case of a problem, e.g., "the shortest-path problem in a specific graph, between two given vertices."
- **Optimization problems:** those finding a legal configuration such that its cost is minimum (or maximum).
  - MST: Given a graph $G=(V, E)$, find the cost of a minimum spanning tree of G.
- An optimization problem Π has instance $I = (F, c)$ where
  - F is the set of *feasible solutions*, and
  - c is a *cost function*, assigning a cost value to each feasible solution $c : F \rightarrow R$
  - The solution of the optimization problem is the feasible solution with optimal (minimal/maximal) cost
- cf., **Optimal** solutions/costs, optimal (**exact**) algorithms (Attn: optimal $\neq$ exact in the theoretic computer science community).

The Traveling Salesman Problem (TSP)

- TSP: Given a set of cities and that distance between each pair of cities, find the distance of a "minimum tour" starts and ends at a given city and visits every city exactly once.



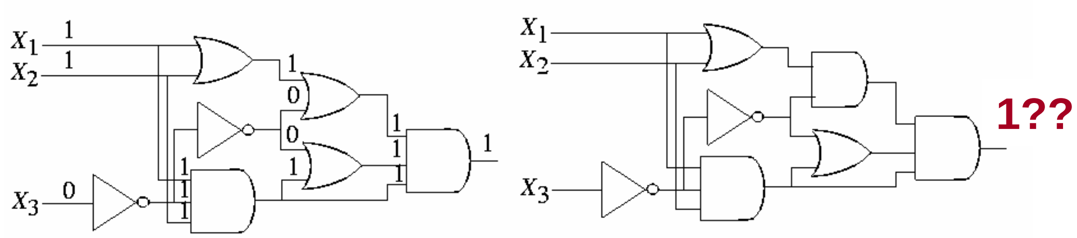


Decision Problem

- **Decision problems:** problem that can only be answered with "yes" or "no"
  - MST: Given a graph $G=(V, E)$ and a bound K, is there a spanning tree with a cost at most K?
  - TSP: Given a set of cities, distance between each pair of cities, and a bound B, is there a route that starts and ends at a given city, visits every city exactly once, and has total distance at most B?
- A decision problem D_{Π} has instances: $I = (F, c, k)$
  - The set of of instances for which the answer is "yes" is given by Y_{Π}.
  - A subtask of a decision problem is *solution checking*: given $f \in F$, checking whether the cost is less than k.
- Could apply binary search on decision problems to obtain solutions to optimization problems.
- NP-completeness is associated with decision problems.

A Decision Problem

- **The Circuit-Satisfiability Problem (Circuit-SAT):**
  - **Instance:** A combinational circuit C composed of AND, OR, and NOT gates.
  - **Question:** Is there an assignment of Boolean values to the inputs that makes the output of C to be 1?
- A circuit is satisfiable if there exists a set of Boolean input values that makes the output of the circuit to be 1.
  - Circuit (a) is satisfiable since $<x_1, x_2, x_3> = <1, 1, 0>$ makes the output to be 1.



(a) satisfiable circuit (b) unsatisfiable circuit

Complexity Class P

- **Complexity class *P*** contains those problems that can be **solved** in polynomial time in the **size of input**.
  - **Input size:** size of encoded "binary" strings.
  - Edmonds: Problems in P are considered **tractable**.
- The computer concerned is a *deterministic Turing machine*
  - *Deterministic* means that each step in a computation is predictable.
  - A *Turing machine* is a mathematical model of a universal computer (any computation that needs polynomial time on a Turing machine can also be performed in polynomial time on any other machine).
- MST is in *P*.

Complexity Class NP

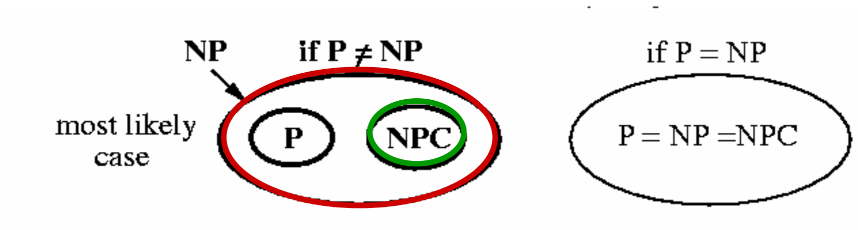
- **Class NP** (**N**ondeterministic **P**olynomial): class of problems that can be **verified** in polynomial time in the size of input.
  - NP: class of problems that can be solved in polynomial time on a nondeterministic machine.
- *Solution checking* can be done in polynomial time on a deterministic machine ⇒ the problem can be solved in polynomial time on a *nondeterministic Turing machine.*
  - *Nondeterministic*: the machine makes a guess, e.g., the right one (or the machine evaluates all possibilities in parallel).
- Is TSP ∈ NP?
  - Need to verify a solution in polynomial time.
    - Guess a tour.
    - Check if the tour visits every city exactly once.
    - Check if the tour returns to the start.
    - Check if total distance $\leq B$.
  - All can be done in $O(n)$ time, so TSP ∈ NP.

Complexity Class NP-Complete

- Still unsettled issue:

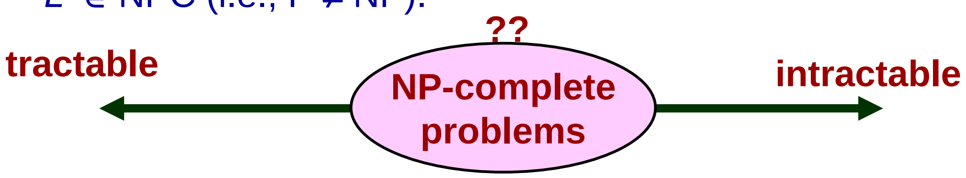
$$P \subset NP \text{ or } P = NP?$$

- There is a strong belief that $P \neq NP$, due to the existence of NP-complete problems.
- **The class NP-complete (NPC):**
  - Developed by S. Cook and R. Karp in early 1970.
  - All problems in NPC have the same degree of difficulty: **Any** NPC problem can be solved in polynomial time $\Rightarrow$ **all** problems in NP can be solved in polynomial time.



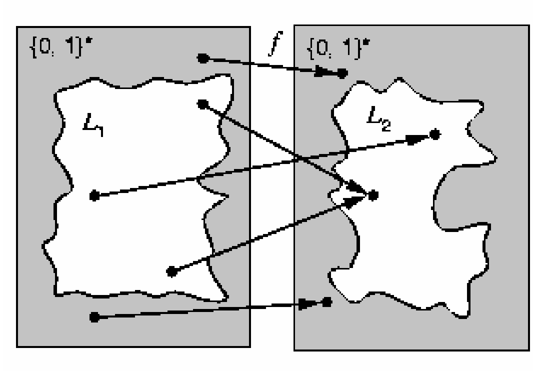
NP-Complete and NP-hard

- **NP-completeness: worst-case** analyses for **decision** problems.
- A **decision** problem L is **NP-complete (NPC)** if
  1. $L \in$ NP, and
  2. $L' \leq_P L$ for every $L' \in$ NP.
- **NP-hard:** If L satisfies property 2, but not necessarily property 1, we say that L is **NP-hard**.
- Suppose $L \in$ NPC.
  - If $L \in P$, then there exists a polynomial-time algorithm for every $L' \in$ NP (i.e., P = NP).
  - If $L \notin P$, then there exists no polynomial-time algorithm for any $L' \in$ NPC (i.e., P ≠ NP).



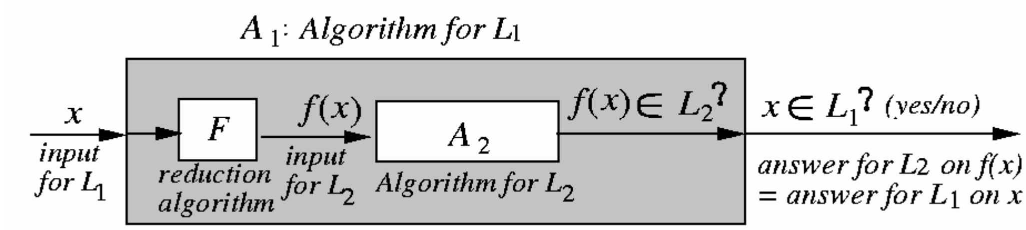
Polynomial-time Reduction

- **Motivation:** Let L_1 and L_2 be two decision problems. Suppose algorithm A_2 can solve L_2. Can we use A_2 to solve L_1?
- **Polynomial-time reduction** f **from** L_1 **to** L_2: $L_1 \leq_P L_2$
  - f reduces input for L_1 into an input for L_2 s.t. the reduced input is a "yes" input for L_2 iff the original input is a "yes" input for L_1.
    - $L_1 \leq_P L_2$ if $\exists$ polynomial-time computable function f: $\{0, 1\}^* \rightarrow \{0, 1\}^*$ s.t. $x \in L_1$ iff $f(x) \in L_2$, $\forall\, x \in \{0, 1\}^*$.
    - **L_2 is at least as hard as L_1.**
  - f is computable in polynomial time.



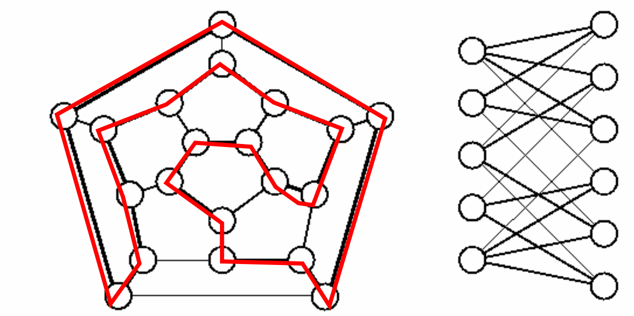
Significance of Reduction

- Significance of $L_1 \leq_P L_2$:
  - $\exists$ polynomial-time algorithm for $L_2 \Rightarrow \exists$ polynomial-time algorithm for L_1 ($L_2 \in$ P $\Rightarrow L_1 \in$ P).
  - $\nexists$ polynomial-time algorithm for $L_1 \Rightarrow \nexists$ polynomial-time algorithm for L_2 ($L_1 \notin$ P $\Rightarrow L_2 \notin$ P).
- $\leq_P$ is transitive, i.e., $L_1 \leq_P L_2$ and $L_2 \leq_P L_3 \Rightarrow L_1 \leq_P L_3$.



Example: HC $\leq_P$ TSP

- The Hamiltonian Circuit Problem (HC)
  - **Instance:** an undirected graph $G = (V, E)$.
  - **Question:** is there a cycle in G that includes every vertex exactly once?
- TSP (The Traveling Salesman Problem)
- How to show HC $\leq_P$ TSP?
  1. Define a function f mapping any HC instance into a TSP instance, and show that f can be computed in polynomial time.
  2. Prove that G has an HC iff the reduced instance has a TSP tour with distance $\leq B$ ($x \in$ HC $\Leftrightarrow f(x) \in$ TSP).

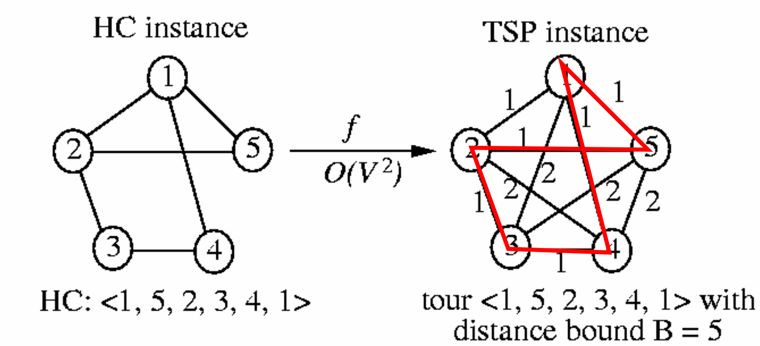


HC $\leq_P$ TSP: Step 1

1. Define a reduction function f for HC $\leq_P$ TSP.
   - Given an arbitrary HC instance $G = (V, E)$ with n vertices
     - Create a set of n cities labeled with names in V.
     - Assign distance between u and v

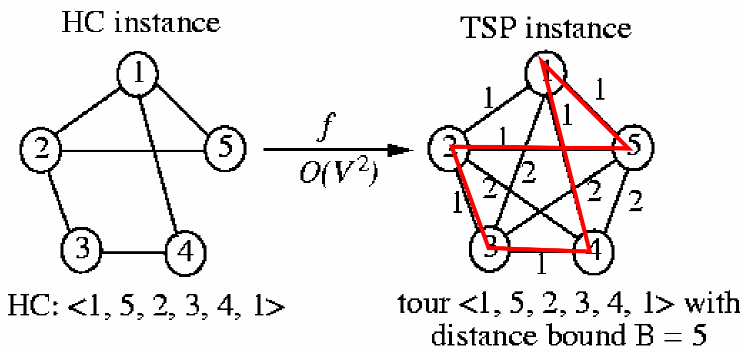
$$d(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E, \\ 2, & \text{if } (u,v) \notin E. \end{cases}$$

     - Set bound $B = n$.
   - f can be computed in $O(V^2)$ time.



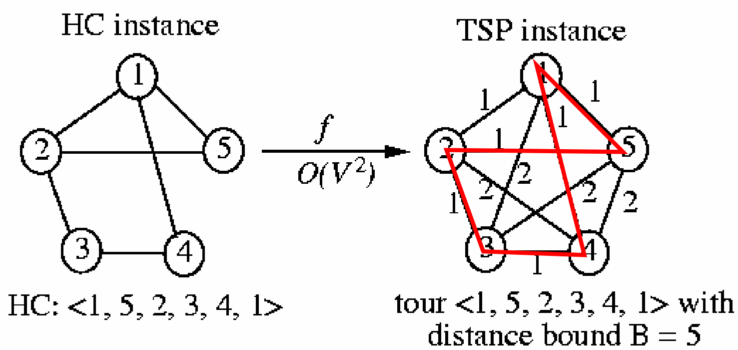
HC $\leq_P$ TSP: Step 2

2. *G* has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $x \in$ HC $\Rightarrow f(x) \in$ TSP.
     - Suppose the HC is $h = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$. Then, h is also a tour in the transformed TSP instance.
     - The distance of the tour h is $n = B$ since there are n consecutive edges in E, and so has distance 1 in $f(x)$.
     - Thus, $f(x) \in$ TSP ($f(x)$ has a TSP tour with distance $\leq B$).



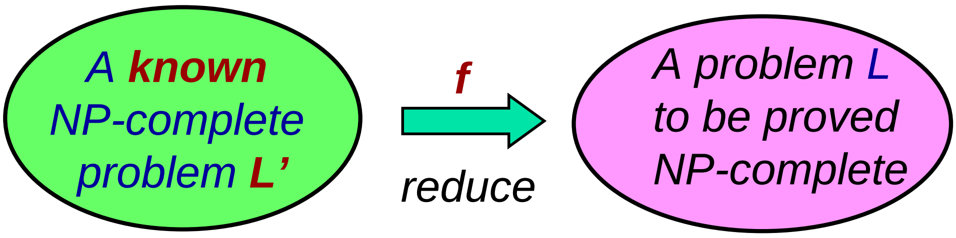
HC $\leq_P$ TSP: Step 2 (cont'd)

2. *G* has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $f(x) \in$ TSP $\Rightarrow x \in$ HC.
     - Suppose there is a TSP tour with distance $\leq n = B$. Let it be $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$..
     - Since distance of the tour $\leq n$ and there are n edges in the TSP tour, the tour contains only edges in E.
     - Thus, $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a Hamiltonian cycle ($x \in$ HC).



Summary: Proving NP-Completeness

- **Five steps for proving that *L* is NP-complete:**
  1. Prove $L \in$ NP.
  2. Select a known NP-complete problem ***L'***.
  3. Construct a reduction *f* transforming **every** instance of ***L'*** to an instance of *L*.
  4. Prove that $x \in L'$ iff $f(x) \in L$ for all $x \in \{0, 1\}^*$.
  5. Prove that *f* is a polynomial-time transformation.
- We have shown that TSP is NP-complete (reducing from HC).



Ch 5. Optimization Algorithms

- Continuous optimization problems
  - Variables are real numbers
- Combinatorial optimization problems
  - Variables are discrete values
  - Useful in EDA

Coping with Optimization Problems

- **Exact solution (may not applicable to big problems)**
  - **Exhaustive search**
    - Feasible only when the problem size is small.
  - **Visit only part of search space**
    - E.g. Branch and bound, dynamic programming, integer linear programming.
- **Approximation algorithms**
  - Guarantee to be a fixed percentage away from the optimum.
  - No general purpose approx. algorithms
  - E.g., MST for the minimum Steiner tree problem.
- **Heuristics**
  - No guarantee of performance
  - E.g. Greedy algorithm, Local search, Tabu search, Simulated annealing (hill climbing), Genetic algorithms, etc.

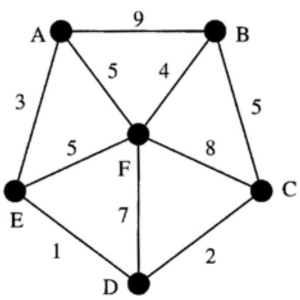
Algorithmic Paradigms

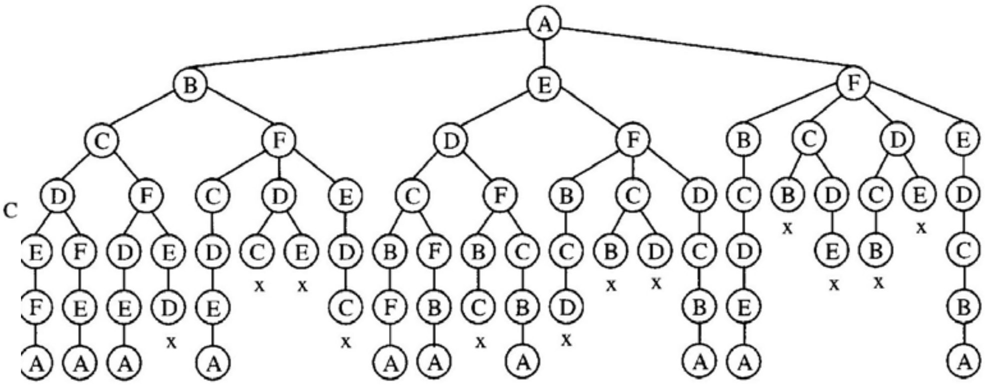
- **Branch and bound:** A search technique with pruning.
- **Mathematical programming:** A system of solving an objective function under constraints.
- **Dynamic programming:** Partition a problem into a collection of sub-problems, the sub-problems are solved, and then the original problem is solved by combining the solutions. (sub-problems are NOT independent)
- **Divide and Conquer:** Partition problems into independent sub-problems.

- **Greedy :** Pick a locally optimal solution at each step.
- **Simulated annealing:** An adaptive, iterative, non-deterministic algorithm that allows “uphill” moves to escape from local optima.
- **Tabu search:** Similar to simulated annealing, but does not decrease the chance of “uphill” moves throughout the search.
- **Genetic algorithm:** A population of solutions is stored and allowed to evolve through successive generations via mutation, crossover, etc.

Exhaustive search

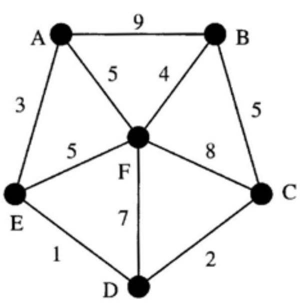
- General principle
  - Systematically assign values to unspecified variables
  - Until a single point in search space is identified , or
  - An implicit constraints makes it impossible to continue
    - backtracking
- Example: TSP
  - X means backtracking

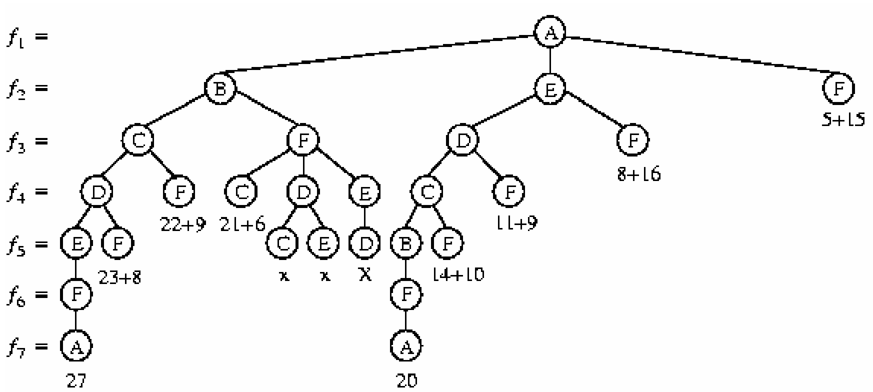




Branch and Bound

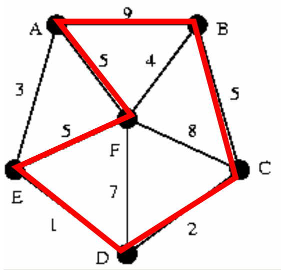
- General principle
  - Estimate the cost lower bound
  - Kill partial solutions higher than the lowest cost
- Example: TSP
  - Use MST as cost lower bound
  - E.g. A→ B→ C→ F = 22; MST of {CDEA} = 6
    - 22+8 > 27 ➔ Killed

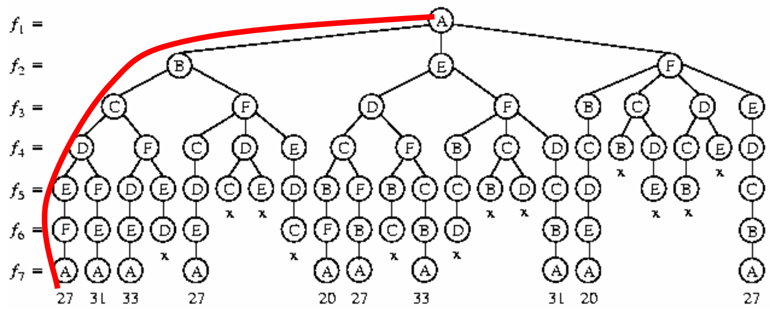




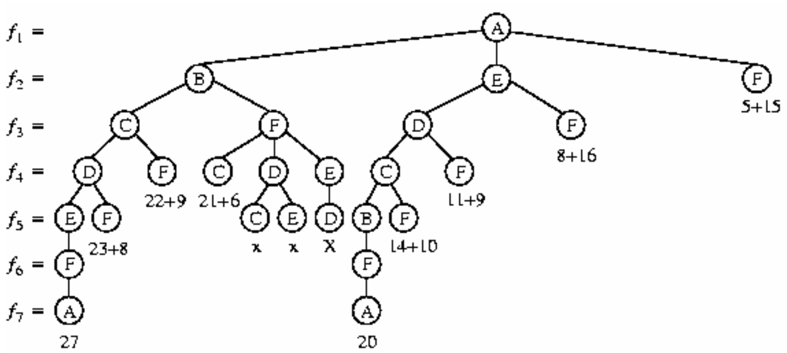
Exhaustive Search vs. Branch and Bound

- TSP example





Backtracking/exhaustive search 72 nodes!



Branch and bound only 27 nodes

Pseudo Code of B&B

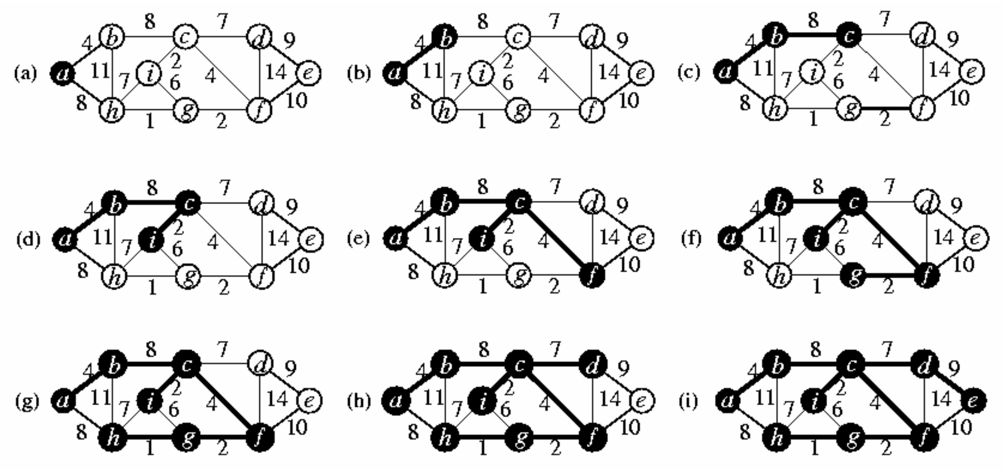
```
float best_cost;
solution_element val[n], best_solution[n];

b_and_b(int k)
{
  float new_cost;
  if (k = n) {
    new_cost := cost(val);
    if (new_cost < best_cost) {
      best_cost := new_cost;
      best_solution := copy(val);
    }
  }
  else if (lower_bound_cost(val,k) ≥ best_cost)
        /* No action, node is killed. */
  else
    for each (el ∈ allowed(val, k)) {
      val[k] := el;
      b_and_b(k + 1);
    }
}

main ()
{
  best_cost := ∞;
  b_and_b(0);
  report(best_solution);
}
```

Figure 5.5 The pseudo-code of the branch-and-bound algorithm.

- Disadvantage:
  - Finding lower bound is not easy all the time

Greedy Algorithms

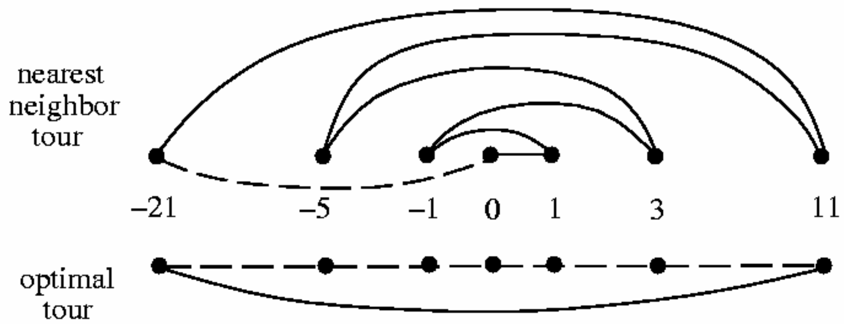
- General principle:
  - Pick a locally optimal solution at each step
- Greedy method does not guarantee performance
  - sometimes is correct: e.g. Prim's algorithm for MST
  - sometimes is not correct : e.g. Nearest neighbor for TSP



Nearest Neighbor for TSP

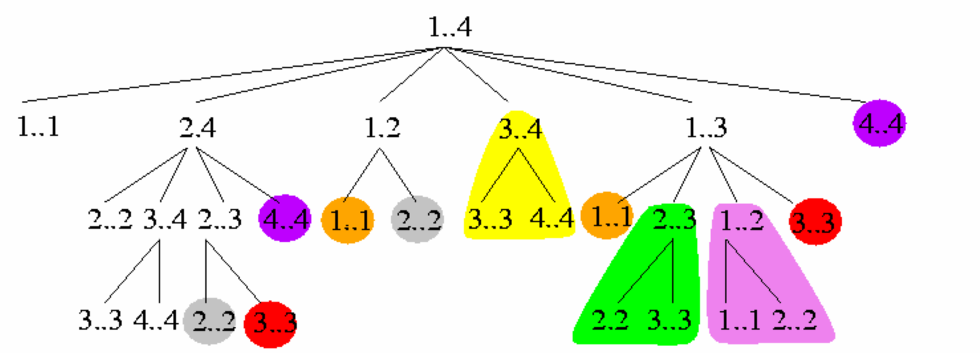
1. pick and visit an initial point p_0;
2. $P \leftarrow p_0$;
3. $i \leftarrow 0$;
4. **while** there are unvisited points **do**
5. visit p_i's closet unvisited point p_{i+1};
6. $i \leftarrow i + 1$;
7. return to p_0 from p_i.

- Simple to implement and very efficient, but **not optimal!**



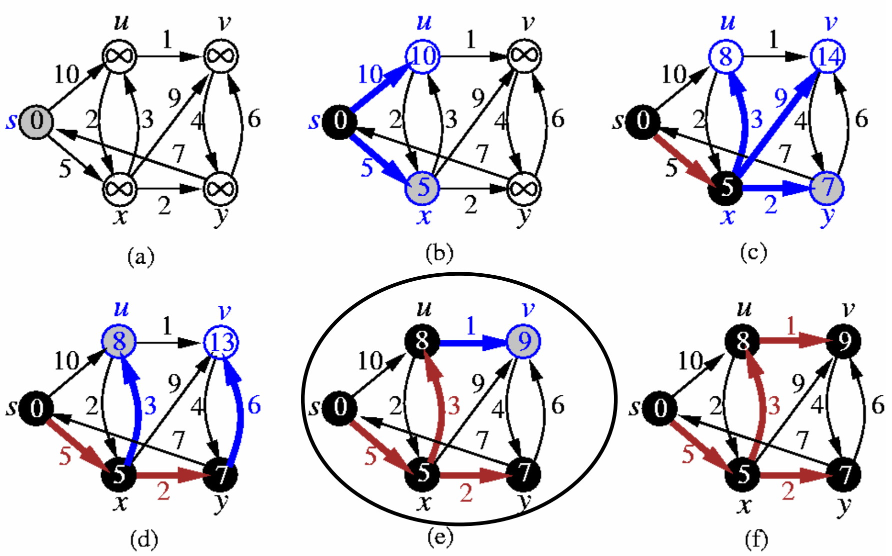
Dynamic Programming (DP) vs. Divide-and-Conquer

- Both solve problems by combining the solutions to subproblems.
- Divide-and-conquer algorithms
  - Partition a problem into **independent** subproblems, solve the subproblems recursively, and then combine their solutions to solve the original problem.
  - Inefficient if they solve the same subproblem more than once.
- Dynamic programming (DP)
  - Applicable when the subproblems are **not independent**.
  - DP solves each subproblem just once.



Dijkstra's Shortest Path

- Reduce search space by dynamic programming
  - Shortest path from s→v = s→u plus u→v
    - Since shortest path s→u is already known (8), calculation is eliminated
    - Shortest path from s→ v is 8+1



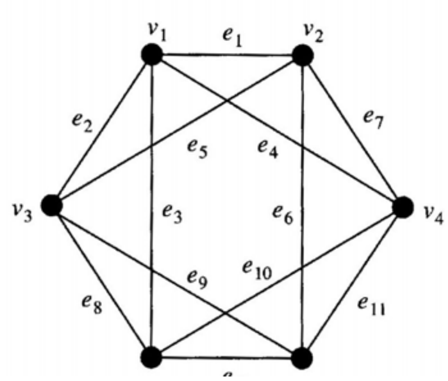
Linear Programming

- General principle
  - Convert problems into the mathematic format
  - Canonical form of LP
    - AX≤b
    - X≥0
- Integer Linear Programming (ILP)
  - Variables are restricted to integers
- 0-1 ILP problems
  - Solutions are restricted to 0,1
- Why ILP useful for EDA?
  - ILP solvers are widely available
  - Problem independent

Example 1: TSP

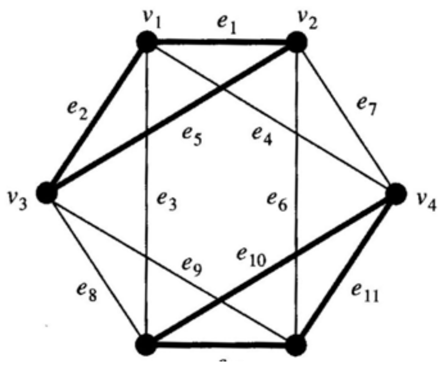
- Variables $x_1 \ldots x_{12}$
  - Xi = 1 means the edge is traveled
  - Xi = 0 means the edge is not traveled
- Minimize cost of travel $\sum_{i=1}^{12} w(e_i)x_i$
- Subject to constraints
  - Every vertex has exactly two edges traveled

$$v_1 : x_1 + x_2 + x_3 + x_4 = 2$$
$$v_2 : x_1 + x_5 + x_6 + x_7 = 2$$
$$v_3 : x_2 + x_5 + x_8 + x_9 = 2$$
$$v_4 : x_4 + x_7 + x_{10} + x_{11} = 2$$
$$v_5 : x_3 + x_8 + x_{10} + x_{12} = 2$$
$$v_6 : x_6 + x_9 + x_{11} + x_{12} = 2$$



Example 1: TSP (cont'd)

- However, not enough constraints
  - Multiple disjoint tour

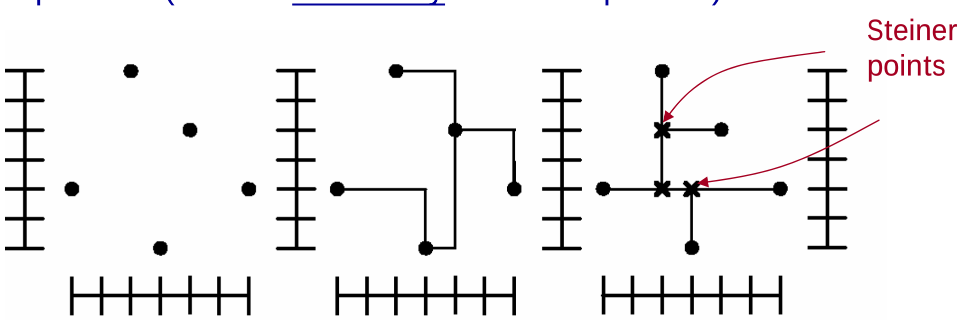


- Add more constraints to avoid multiple disjoint tour

$$\{v_1, v_2, v_3\}\ \{v_4, v_5, v_6\} : x_3 + x_4 + x_6 + x_7 + x_8 + x_9 \geq 2$$
$$\{v_1, v_3, v_5\}\ \{v_2, v_4, v_6\} : x_1 + x_4 + x_5 + x_9 + x_{10} + x_{12} \geq 2$$
$$\{v_1, v_2, v_4\}\ \{v_3, v_5, v_6\} : x_2 + x_3 + x_5 + x_6 + x_{10} + x_{11} \geq 2$$

Approximation Algorithm
Spanning Tree vs. Steiner Tree

- **Manhattan distance:** If two points (nodes) are located at coordinates (x_1, y_1) and (x_2, y_2), the Manhattan distance between them is given by $d_{12} = |x_1 - x_2| + |y_1 - y_2|$.
- **Rectilinear spanning tree:** a spanning tree that connects its nodes using Manhattan paths (Fig. (b) below).
- **Steiner tree:** a tree that connects its nodes, and additional points (Steiner points) are permitted to used for the connections.
- The minimum rectilinear spanning tree problem is in P, while the minimum rectilinear Steiner tree (Fig. (c)) problem is NP-complete.
  - The spanning tree algorithm can be an *approximation* for the Steiner tree problem (at most 50% away from the optimum).



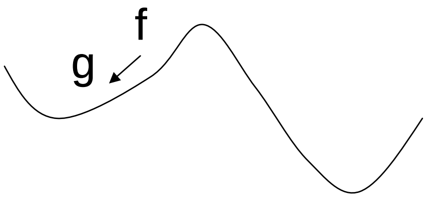
(a) (b) (c)

Local Search

- General principle
  - Search only the neighbors of the current solution
  - Move to the neighbor with lower cost than the current solution
- Problem
  - Can be trapped in local minimum

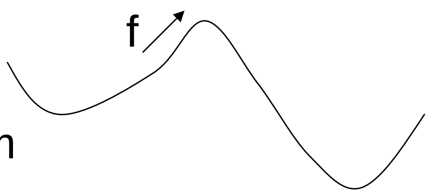
```
local_search()
{
  struct feasible_solution f;
  set of struct feasible_solution G;

  f ← initial_solution();
  do {
     G ← {g|g ∈ N(f), c(g) < c(f)};
     if (G ≠ ∅)
        f ← "any element of G";
  } while (G ≠ ∅);
  "report f";
}
```

Figure 5.8 The pseudo-code description of local search.



Simulated Annealing (SA)

- General Principle: mimic material cooling process
  - Search the neighbors of current solution
  - A good move means the cost decreases
    - always accepted
  - A bad move means the cost increases
    - Accepted with probability exp (-Δcost/T)
- Analogy
  - Energy = cost function
  - Temperature = controlling parameter T
- Advantage: can escape local minimum
  - 'Uphill climbing' is possible



Pseudo Code of SA

```
int accept(struct feasible_solution f, g)
{
  float Δc;

  Δc ← c(g) − c(f);
  if (Δc ≤ 0)
    return 1;
  else return (e^(−Δc/T) > random(1));
}

simulated_annealing()
{
  struct feasible_solution f, g;
  float T;

  f ← initial_solution();
  do {
     do {
        g ← "some element of N(f)";
        if (accept(f, g))
          f ← g
     while (!thermal_equilibrium());
     T ← new_temperature(T);
  while (!stop());
  "report f";
}
```


Tabu Search

- General Principle: avoid staying in the 'taboo' solutions
  - Keep a list of visited solutions : Taboo list
  - Always move to a new solution other than the taboo list
    - even the new solution is poor than the current solution

```
tabu_search()
{
  struct feasible_solution f, g, b;
  set of struct feasible_solution G;
  "k-element FIFO queue of" feasible_solution Q;

  Q ← "empty";
  b ← initial_solution();
  f ← initial_solution();
  do {
     G ← "some subset of N(f) such that ∀s ∈ Q, s ∉ G";
     if (G ≠ ∅) {
       g ← "cheapest element of G";
       "shift g into Q";
       f ← g;
       if (c(f) < c(b))
         b ← f;
     }
  }
  while (G ≠ ∅ or stop());
```

Genetic Algorithms (GA)

- General Principle
  - Survival of the fittest
  - Keep a group of feasible solutions
    - 'population'
  - 'Parent' population generates the 'child' population
    - Keep only the best children

Important steps in GA

- Cross over: Two feasible solutions generate their child by switching chromosomes

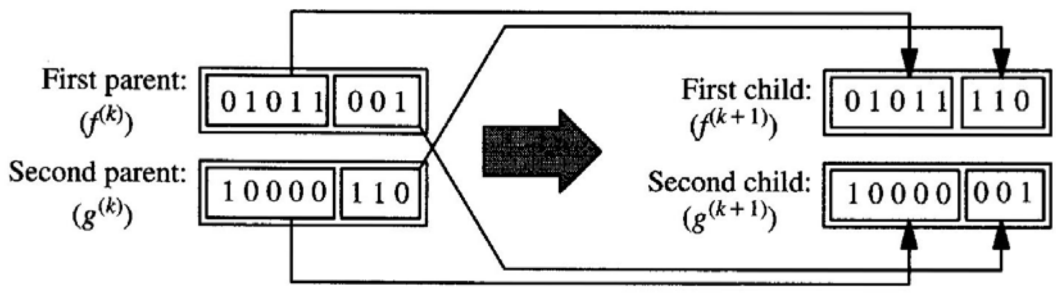


Figure 5.11 The generation of a pair of children by crossover.

- Mutation: some chromosomes can change by probability

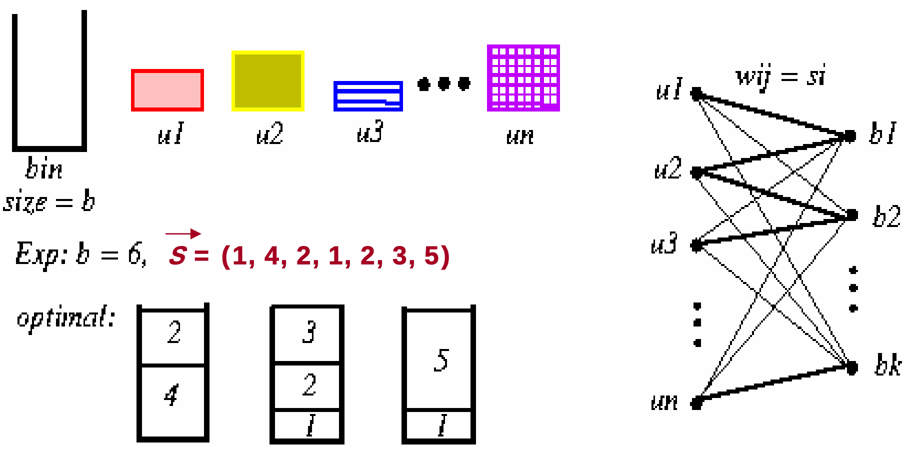
Pseudo Code of GA

```
genetic()
{
  int pop_size;
  set of struct chromosome pop, newpop;
  struct chromosome parent1, parent2, child;

  pop ← ∅;
  for (i ← 1; i ≤ pop_size; i ← i + 1)
    pop ← pop ∪ {"chromosome of random feasible solution"};
  do {
     newpop ← ∅;
     for (i ← 1; i ≤ pop_size; i ← i + 1) {
       parent1 ← select(pop);
       parent2 ← select(pop);
       child ← crossover(parent1, parent2);
       newpop ← newpop ∪ {child};
     }
     pop ← newpop;
  } while (!stop());
  "report best solution";
}
```

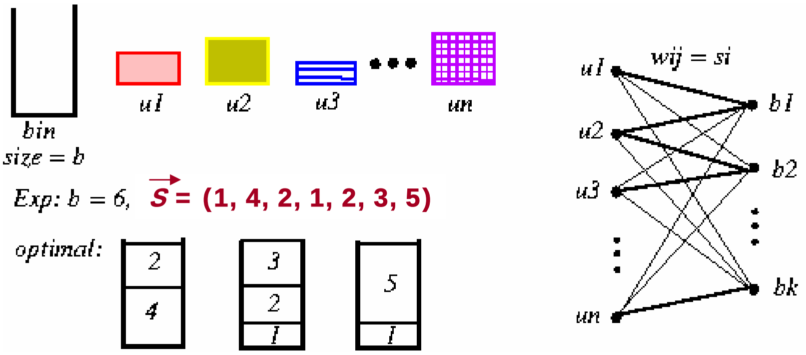
Figure 5.12 The pseudo-code description of a genetic algorithm.

Example 2: Bin Packing

- **The Bin-Packing Problem** Π **:** Items $U = \{u_1, u_2, \ldots, u_n\}$, where u_i is of an integer size s_i; set B of bins, each with capacity b.
- **Goal:** Pack all items, minimizing # of bins used. (**NP-hard!**)



Example 2: Bin Packing (cont'd)



- Greedy approximation alg.: First-Fit Decreasing (FFD)
  - *FFD*(Π) ≤ 11OPT(Π)/9 + 4)
- Use **integer linear programming (ILP)** to find a solution using |*B*| bins, then search for the smallest feasible |*B*|.

Example 2: Bin Packing (cont'd)

- 0-1 variable: x_{ij}=1 if item u_i is placed in bin b_j, 0 otherwise.

$$
\begin{aligned}
\text{max} \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\
\textit{subject to} \quad & \\
& \sum_{\forall i \in U} w_{ij}x_{ij} \leq b_j, \forall j \in B \quad /* \textit{ capacity constraint } */ \quad (1) \\
& \sum_{\forall j \in B} x_{ij} = 1, \forall i \in U \quad /* \textit{ assignment constraint } */ \quad (2) \\
& \sum_{ij} x_{ij} = n \quad /* \textit{ completeness constraint } */ \quad (3) \\
& x_{ij} \in \{0, 1\} \quad /* \text{ 0, 1 } \textit{constraint } */ \quad (4)
\end{aligned}
$$

- **Step 1:** Set |*B*| to the lower bound of the # of bins.
- **Step 2:** Use the ILP to find a feasible solution.
- **Step 3:** If the solution exists, the # of bins required is |*B*|. Then exit.
- **Step 4:** Otherwise, set |*B*| ← |*B*| + 1. Goto Step 2.