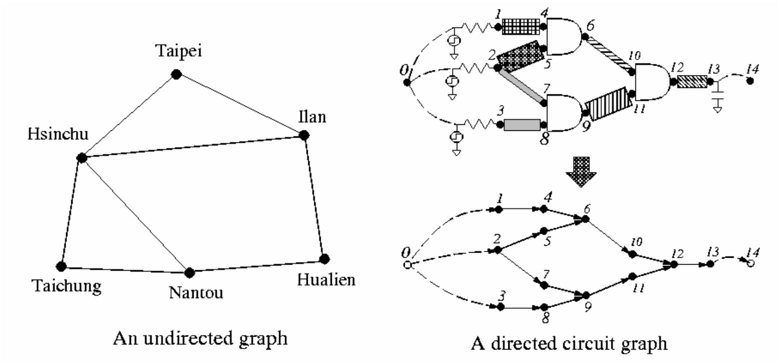
Unit 2: Algorithmic Graph Theory

- Course contents:
  - Introduction to graph theory
  - Basic graph algorithms
- Reading
  - Chapter 3
  - Reference: Cormen, Leiserson, and Rivest, *Introduction to Algorithms*, 2nd Ed., McGraw Hill/MIT Press, 2001.

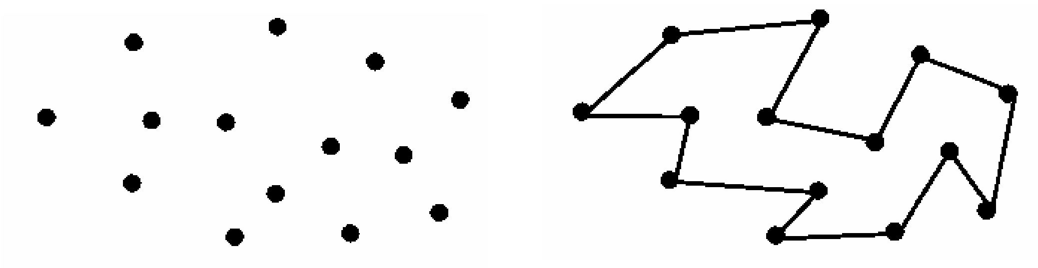


Algorithms

- **Algorithm:** A well-defined procedure for transforming some **input** to a desired **output**.
- **Major concerns:**
  - **Correctness:** Does it **halt**? Is it **correct**?
  - **Efficiency:** **Time** complexity? **Space** complexity?
    - Worst case? Average case? (Best case?)
- **Better algorithms?**
  - **How: Faster** algorithms? Algorithms with **less space** requirement?
  - **Optimality:** Prove that an algorithm is **best possible/optimal**? Establish a **lower bound**?

Example: Traveling Salesman Problem (TSP)

- **Instance:** A set of points (cities) P together with a distance $d(p, q)$ between any pair $p, q \in P$.
- **Output:** What is the shortest circular route that starts and ends at a given point and visits all the points.

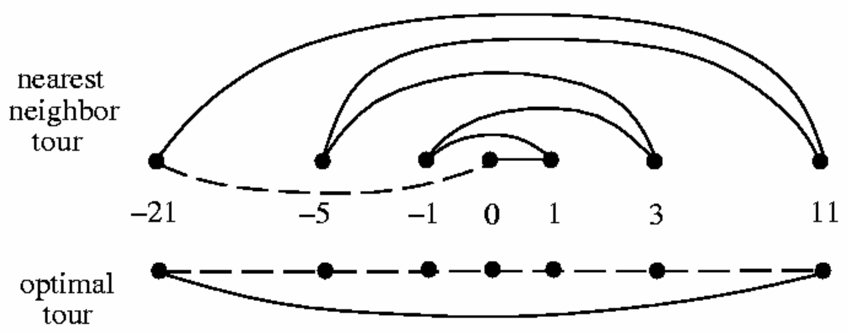


- Correct and efficient algorithms?

Nearest Neighbor Tour

1. pick and visit an initial point p_0;
2. $P \leftarrow p_0$;
3. $i \leftarrow 0$;
4. **while** there are unvisited points **do**
5.   visit p_i's closet unvisited point p_{i+1};
6. $i \leftarrow i + 1$;
7. return to p_0 from p_i.

- Simple to implement and very efficient, but **incorrect!**



A Correct, but Inefficient Algorithm

```
d ← ∞ ;
for each of the n! permutations π_i of the n points
    if (cost(π_i) ≤ d) then
        d ← cost(π_i);
        T_min ← π_i;
return T_min.
```

- **Correctness:** Tries all possible orderings of the points ⇒ Guarantees to end up with the shortest possible tour.
- **Efficiency:** Tries *n!* possible routes!
  - 120 routes for 5 points, 3,628,800 routes for 10 points, 20 points?
- No known efficient, correct algorithm for TSP!

Example: Sorting

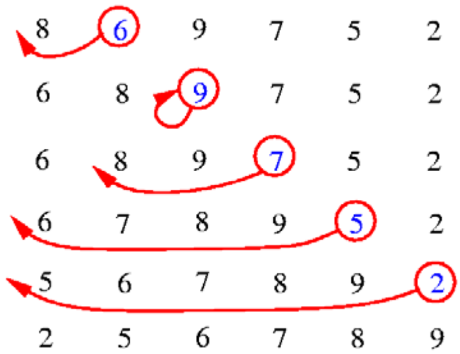
- **Instance:** A sequence of n numbers $<a_1, a_2, \ldots, a_n>$.
- **Output:** A permutation $<a_1', a_2', \ldots, a_n'>$ such that $a_1' \leq a_2' \leq \ldots \leq a_n'$.

  Input: <8, 6, 9, 7, 5, 2, 3>

  Output: <2, 3, 5, 6, 7, 8, 9 >
- Correct and efficient algorithms?

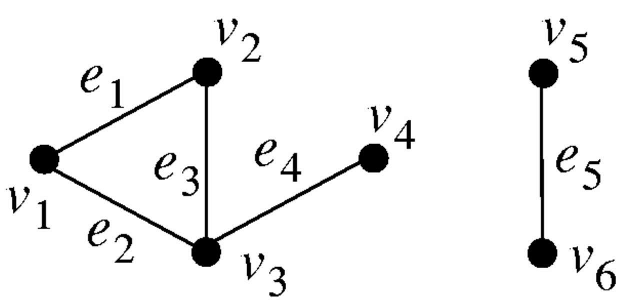
Insertion Sort

```
InsertionSort(A)
for j ← 2 to length[A] do
    key ← A[j];
    /* Insert A[j] into the sorted sequence A[1..j-1]. */
    i ← j - 1;
    while i > 0 and A[i] > key do
        A[i+1] ← A[i];
        i ← i - 1;
    A[i+1] ← key;
```



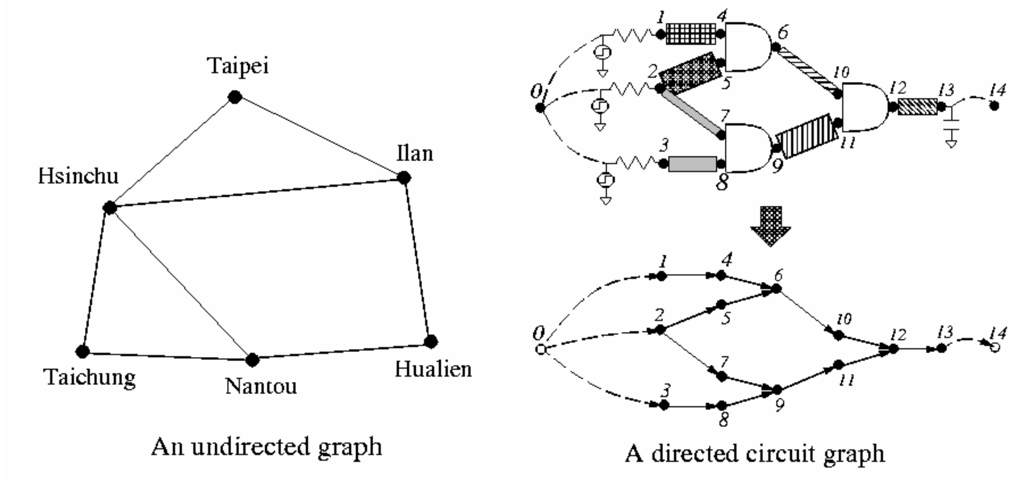
Graph

- Graph: A mathematical object representing a set of "points" and "interconnections" between them.
- A **graph** *G* = (*V, E*) consists of a set *V* of **vertices** (**nodes**) and a set *E* of **directed** or **undirected edges**.
  - *V* is the vertex set: $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, |V|=6
  - *E* is the edge set: $E = \{e_1, e_2, e_3, e_4, e_5\}$, |E|=5
  - An edge has two endpoints, e.g. $e_1 = (v_1, v_2)$
  - For simplicity, use *V* for |*V*| and *E* for |*E*|.

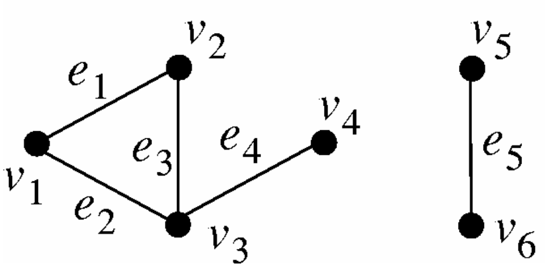


Example Graphs

- Any binary relation is a graph.
  - Network of roads and cities
  - Circuit representation



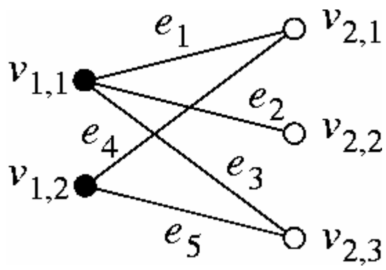
Terminology



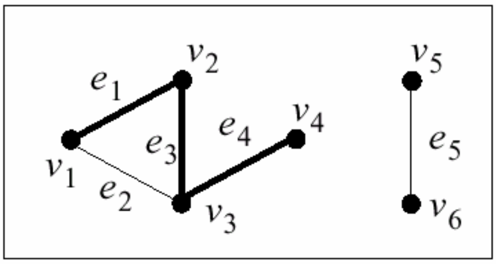
- Degree of a vertex: degree(v_3) = 3, degree(v_2) = 2
- Subgraph of a graph:
- Complete (sub)graph: $V' = \{v_1, v_2, v_3\}$, $E' = \{e_1, e_2, e_3\}$
- (Maximal/maximum) clique: maximal/maximum complete subgraph
- Selfloop
- Parallel edges
- Simple graph
- Multigraph

Terminology (cont'd)

- Bipartite graph $G = (V_1, V_2, E)$
- Path
- Cycle: a closed path
- Connected vertices
- Connected graph
- Connected components



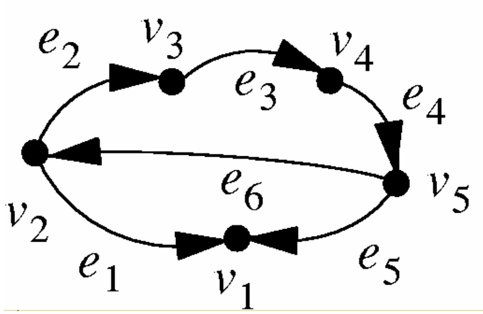
A bipartite graph



Path $p = <v_1, v_2, v_3, v_4>$
Cycle $C = <v_1, v_2, v_3, v_1>$

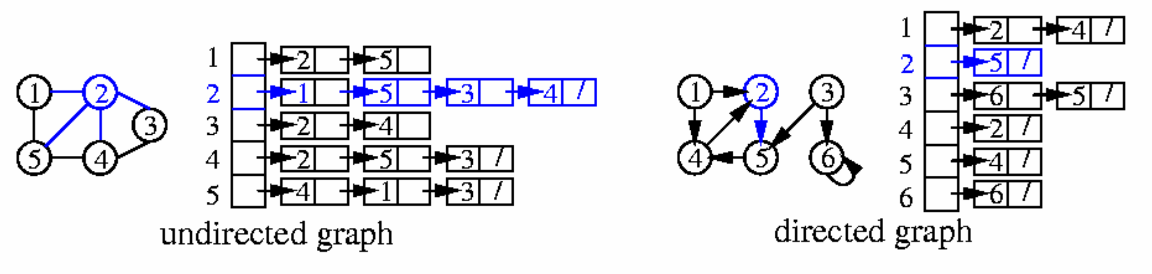
Terminology (cont'd)

- Weighted graph:
  - Edge weighted and/or vertex weighted
- Directed graph: edges have directions
  - Directed path
  - Directed cycle
  - Directed acyclic graph (DAG)
  - In-degree, out-degree
  - Strongly connected vertices
    - Strongly connected components {v1}{v2, v3, v4, v5}
  - Weekly connected vertices



Graph Representation: Adjacency List

- **Adjacency list:** An array *Adj* of |*V* | lists, one for each vertex in *V*. For each *u* ∈ *V*, *Adj*[*u*] pointers to all the vertices adjacent to *u*.
- Advantage: *O*(*V*+*E*) storage, good for **sparse** graph.
- Drawback: Need to traverse list to find an edge.

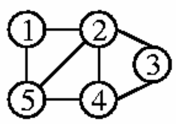


Graph Representations: Adjacency Matrix

- **Adjacency matrix:** A |*V*| × |*V*| matrix *A* = (a_{ij}) such that

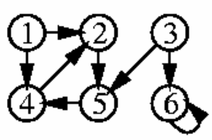
$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Advantage: *O*(1) time to find an edge.
- Drawback: *O*(V^2) storage,suitable for **dense** graph.
- How to save space if the graph is undirected?



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

undirected graph

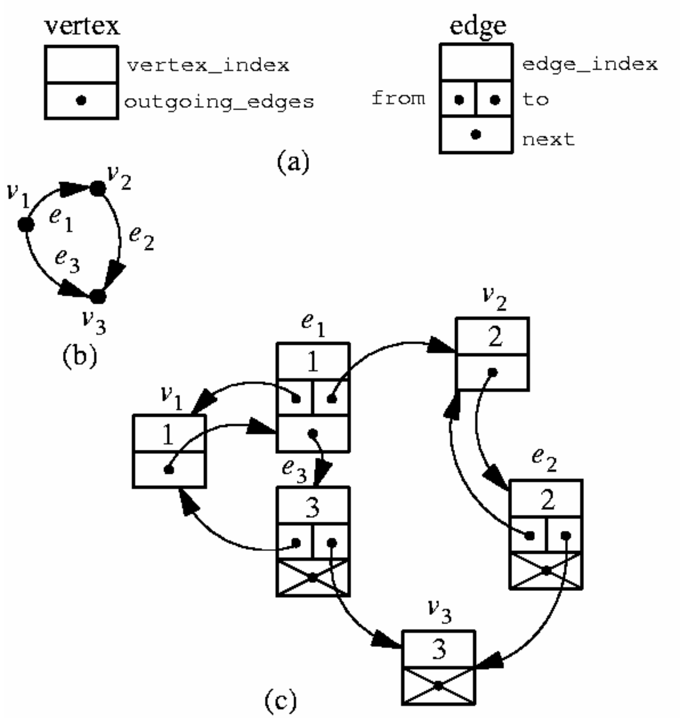


	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

directed graph

Explicit Edges and Vertices

```
struct vertex {
  int vertex_index;
  struct edge *outgoing_edges;
};

struct edge {
  int edge_index;
  struct vertex *from, *to;
  struct edge *next;
};
```



Tradeoffs between Adjacency List and Matrix

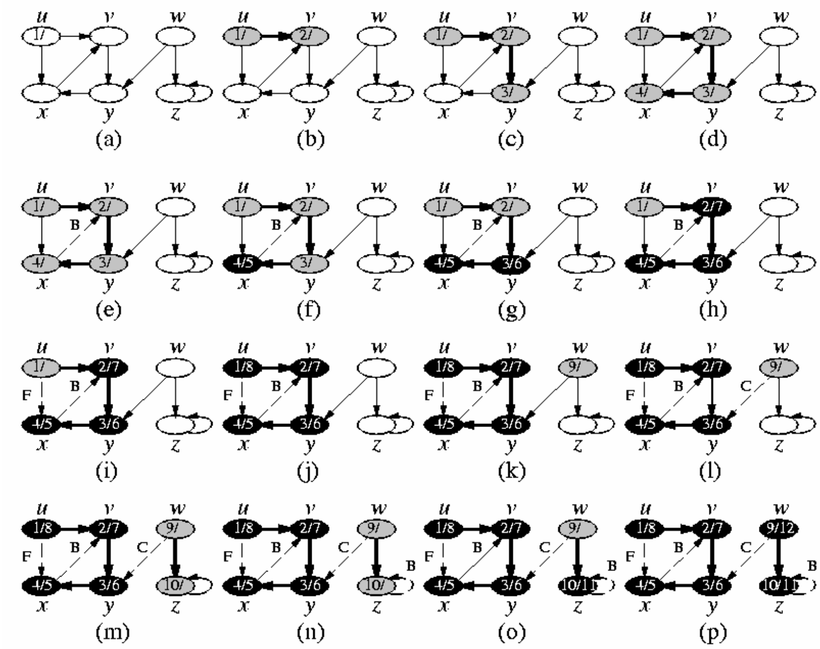
Comparison	Winner
Faster to find an edge?	matrix
Faster to find vertex degree?	list
Faster to traverse the graph?	list $O(V+E)$ vs. matrix $O(V^2)$
Storage for sparse graph?	list $O(V+E)$ vs. matrix $O(V^2)$
Storage for dense graph?	matrix (small win)
Edge insertion or deletion?	matrix $O(1)$
Weighted-graph implementation?	?
Better for most applications?	list

Depth-First Search (DFS) [Cormen]

```
DFS(G)
for each vertex u ∈ V[G]
    color[u] ← WHITE;
    π [u] ←NIL;
time ← 0;
for each vertex u ∈ V[G]
    if color[u] = WHITE
        DFS-Visit(u).

DFS-Visit(u)
color[u] ← GRAY;
 /* White vertex u has just been
    discovered. */
d[u] ← time ← time +1;
for each vertex v ∈ Adj[u]
    /* Explore edge (u,v). */
    if color[v] = WHITE
        π [v] ← u;
        DFS-Visit(v);
color[u] ← BLACK;
  /* Blacken u; it is finished. */
f[u] ← time ← time+1.
```

- *color*[*u*]:
  white (undiscovered) → gray (discovered) → black (explored: out edges are all discovered)
- *d*[*u*]: discovery time (gray)
- *f*[*u*]: finishing time (black)
- π[*u*]: predecessor
- Time complexity: *O*(*V*+*E*) (adjacency list).

DFS Example [Cormen]



- *color*[*u*]: white → gray → black.
- Depth-first **forest**: $G_\pi = (V, E_\pi)$, $E_\pi = \{(\pi[v], v) \in E \mid v \in V, \pi[v] \neq \text{NIL}\}$
  - {u→ v→x→y} {w→z}

DFS Pseudo Code in Text

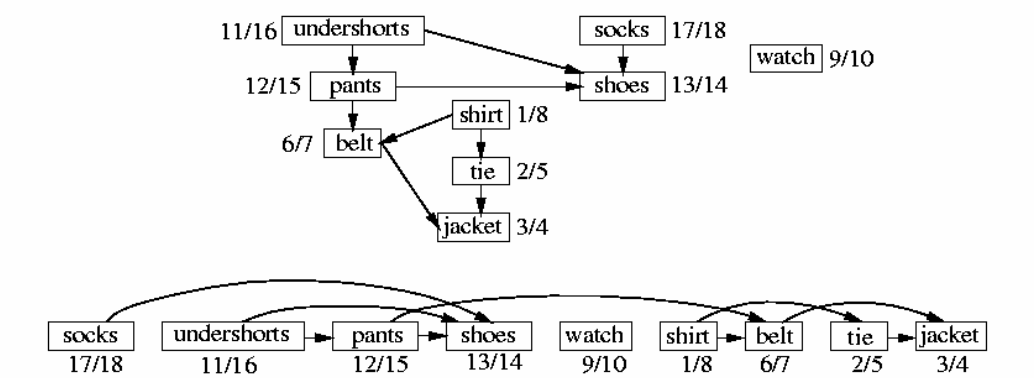
```
/* Given is the graph G(V, E) */

struct vertex {
   . . .
  int mark;
};

dfs(struct vertex v)
{
   v.mark ← 0;
   "process v";
   for each (v, u) ∈ E {
     "process (v, u)";
     if (u.mark)
        dfs(u);
   }
}

main ()
{
  for each v ∈ V
     v.mark ← 1;
  for each v ∈ V
     if (v.mark)
        dfs(v);
}
```


DFS Application 1: Topological Sort

- A **topological sort** of a directed acyclic graph (DAG) $G = (V, E)$ is a linear ordering of V s.t. $(u, v) \in E \Rightarrow u$ appears before v.

> Topological-Sort(G)
> 1. call DFS(G) to compute finishing times $f[v]$ for each vertex v
> 2. as each vertex is finished, insert it onto the front of a linked list
> 3. **return** the linked list of vertices

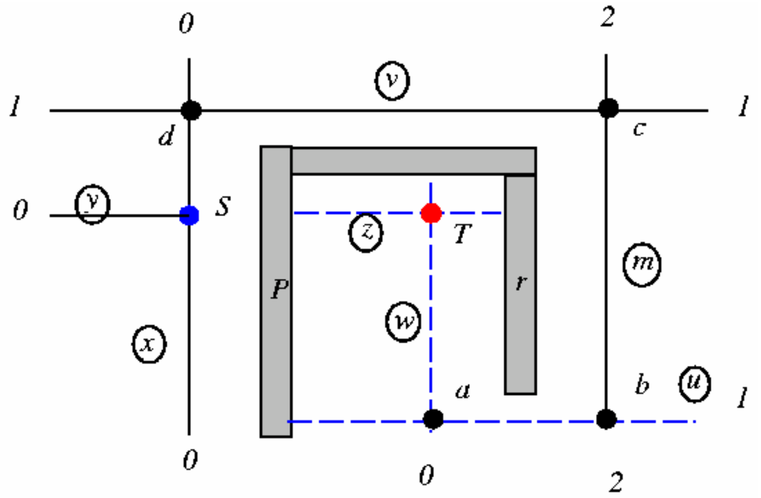
- Time complexity: $O(V+E)$ (adjacent list).



Vertices are arranged from left to right in order of decreasing finishing times.

DFS Application 2: Hightower's Maze Router

- A single escape point on each line segment.
- If a line parallels to the blocked cells, the escape point is placed just past the endpoint of the segment.
- Time and space complexities: $O(L)$, where L is the # of line segments generated.

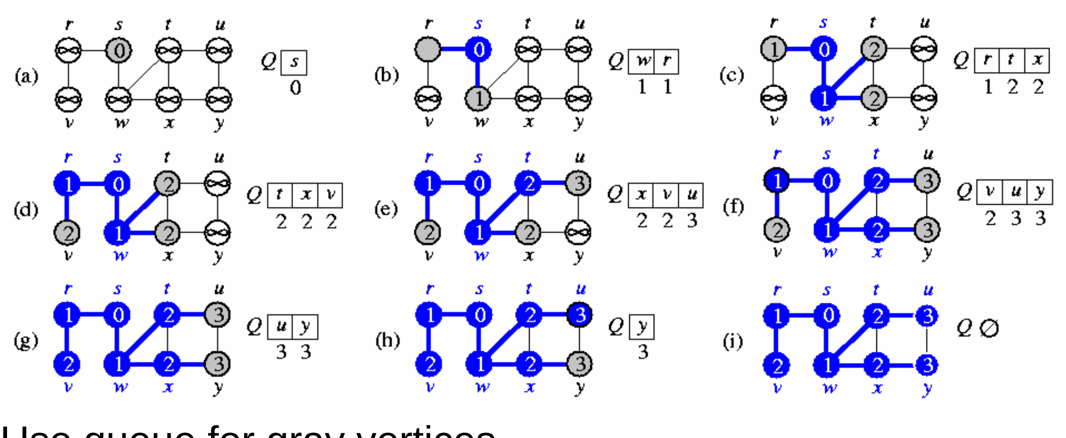


Breadth-First Search (BFS) [Cormen]

```
BFS(G,s)
for each vertex u ∈ V[G]-{s}
    color[u] ←WHITE;
    d[u] ← ∞;
    π [u] ← NIL;
color[s] ←GRAY;
d[s] ← 0;
π[s] ←NIL;
Q ← {s};
while Q ≠ ∅
    u ← head[Q];
    for each vertex v ∈ Adj[u]
        if color[v] = WHITE
            color[v] ←GRAY;
            d[v] ← d[u]+1;
            π [v] ← u;
            Enqueue(Q,v);
    Dequeue(Q);
    color[u] ←BLACK}.
```

- *color*[*u*]:
  white (undiscovered) →
  gray (discovered) →
  black (explored: out edges are all discovered)
- *d*[*u*]: distance from source *s*
- π[*u*]: predecessor of *u*
- Use queue for gray vertices
- Time complexity: $O(V+E)$ (adjacency list).

BFS Example [Cormen]



- Use queue for gray vertices.
  - Each vertex is enqueued and dequeued once: *O*(*V*) time.
  - Each edge is considered once: *O*(*E*) time.
- Breadth-first tree:
  - $G_\pi = (V_\pi, E_\pi)$, $V_\pi = \{v \in V \mid \pi[v] \neq \text{NIL}\} \cup \{s\}$
    - {s, w, r, t, x, v, u, y}
  - $E_\pi = \{(\pi[v], v) \in E \mid v \in V_\pi - \{s\}\}$.
    - {(s,w), (s,r), (w,t), (w,x), (r,v), (t,u), (x,y)}

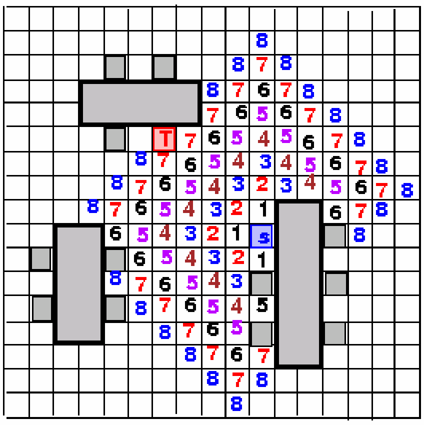
BFS Pseudo Code in Text

```
main ()
{
    for each v ∈ V
        v.mark ← 1;
    for each v ∈ V
        if (v.mark) {
            v.mark ← 0;
            bfs(v);
        }
}
```

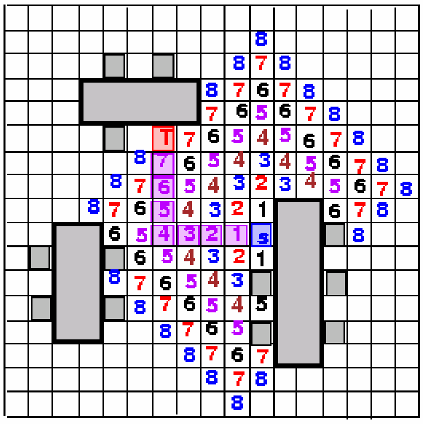
```
bfs(struct vertex v)
{
    struct fifo *Q;
    struct vertex u, w;
    Q ← ();
    shift_in(Q, v);
    do { w ← shift_out(Q);
          "process w";
          for each (w, u) ∈ E {
              "process (w, u)";
              if (u.mark) {
                  u.mark ← 0;
                  shift_in(Q, u);
              }
          }
    } while (Q ≠ ())
}
```

BFS Application: Lee's Maze Router

- Find a path from *S* to *T* by "wave propagation."
- Discuss mainly on single-layer routing
- Strength: Guarantee to find a minimum-length connection between 2 terminals if it exists.
- Weakness: Time & space complexity for an $M \times N$ grid: $O(MN)$ (huge!)



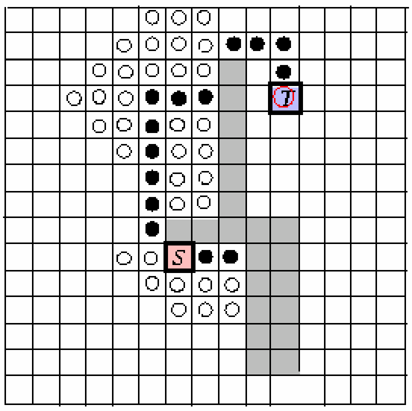
Filing



Retrace

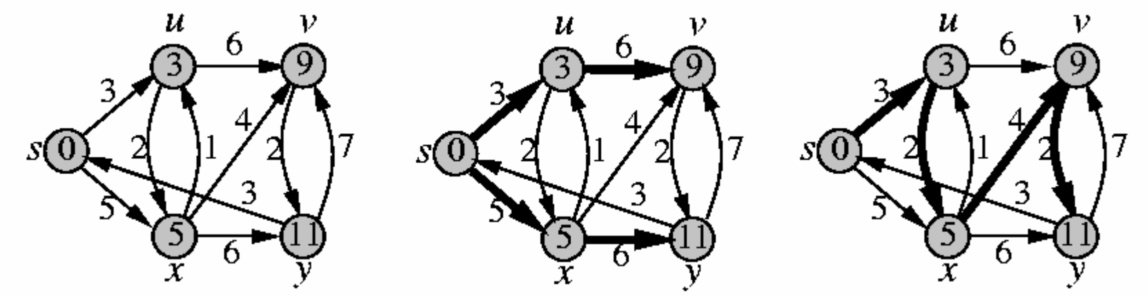
BFS + DFS Application: Soukup's Maze Router

- Depth-first (line) search is first directed toward target *T* until an obstacle or *T* is reached.
- Breadth-first (Lee-type) search is used to "bubble" around an obstacle if an obstacle is reached.
- Time and space complexities: $O(MN)$, but 10--50 times faster than Lee's algorithm.
- Find *a* path between *S* and *T*, but may not be the shortest!



Shortest Paths (SP)

- **The Shortest Path (SP) Problem**
  - **Given:** A **directed** graph *G=(V, E)* with edge weights, and a specific **source node** *s*.
  - **Goal:** Find a minimum weight path (or cost) from *s* to every other node in *V*.
- Applications: weights can be distances, times, wiring cost, delay. etc.
- **Special case:** BFS finds shortest paths for the case when all edge weights are 1.

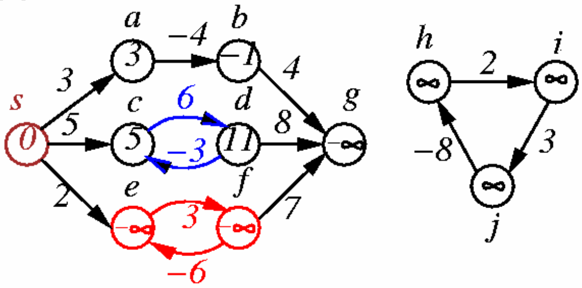


Weighted Directed Graph

- A weighted, directed graph *G* = (*V*, *E*) with the weight function *w*: *E* → R.
  - Weight of path *p* = <$v_0, v_1, \ldots, v_k$>: $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$.
  - **Shortest-path weight** from *u* to *v*, δ(*u*, *v*):

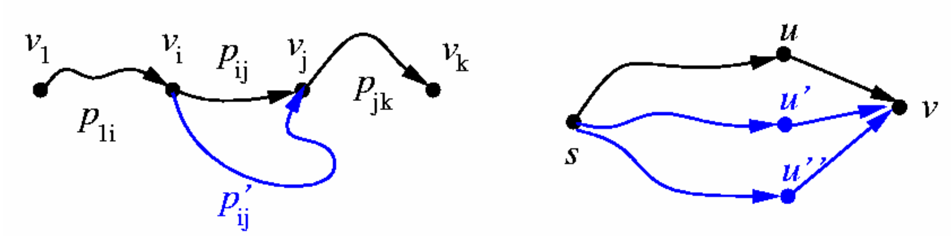
$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\} & \text{if there is a path from } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

- **Warning!** negative-weight edges/cycles are a problem.
  - Cycle <*e, f, e*> has weight -3 < 0 ⇒ δ(*s, g*) = - ∞.
  - Vertices *h, i, j* not reachable from *s* ⇒ δ(*s, h*) = δ(s, i) = δ(*s, j*) = ∞.
- Algorithms apply to the cases for negative-weight edges/cycles??



Optimal Substructure of a Shortest Path

- Subpaths of shortest paths are shortest paths.
  - Let $p = \langle v_1, v_2, \ldots, v_k \rangle$ be a shortest path from vertex v_1 to vertex v_k, and $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j, $1 \le i \le j \le k$. Then, p_{ij} is a shortest path from v_i to v_j. (NOTE: reverse is not necessarily true!)
- Suppose that a shortest path p from a source s to a vertex v can be decomposed into $s \overset{p'}{\leadsto} u \to v$. Then, $\delta(s, v) = \delta(s, u) + w(u, v)$.
- For all edges $(u, v) \in E$, $\delta(s, v) \le \delta(s, u) + w(u, v)$.

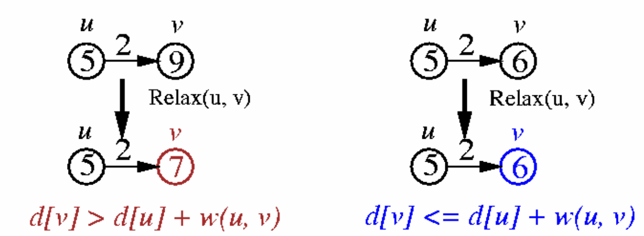


subpaths of shortest paths

Relaxation

```
Initialize-Single-Source(G, s)
1. for each vertex v ∈ V[G]
2.     d[v] ← ∞;
       /* upper bound on the weight of a shortest path from s to v */
3.     π[v] ←NIL; /* predecessor of v */
4. d[s] ← 0;

Relax(u, v, w)
1. if d[v] > d[u]+w(u, v)
2.     d[v] ← d[u]+w(u, v);
3.     π[v] ← u;
```

- $d[v] \le d[u] + w(u, v)$ after calling Relax(u, v, w).
- $d[v] \ge \delta(s, v)$ during the relaxation steps; once $d[v]$ achieves its lower bound $\delta(s, v)$, it never changes.
- Let $s \leadsto u \to v$ be a shortest path. If $d[u] = \delta(s, u)$ prior to the call Relax(u, v, w), then $d[v] = \delta(s, v)$ after the call.



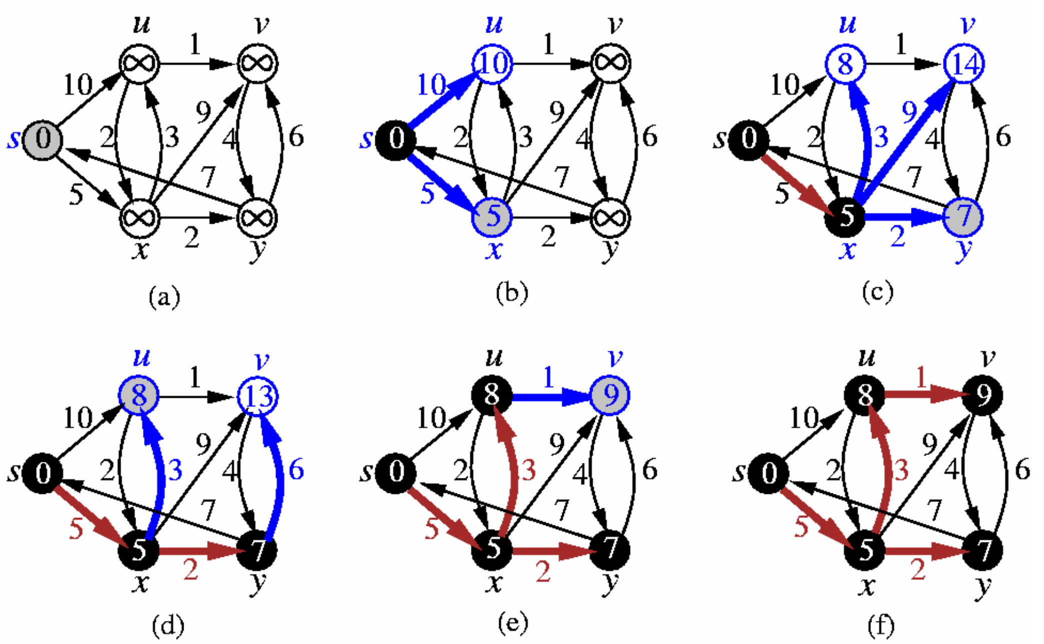
Dijkstra's Shortest-Path Algorithm

Dijkstra(G, w, s)
1. Initialize-Single-Source(G, s);
2. $S \leftarrow \emptyset$;
3. $Q \leftarrow V[G]$;
4. **while** $Q \neq \emptyset$
5. $\quad u \leftarrow$ Extract-Minimum-Element(Q);
6. $\quad S \leftarrow S \cup \{u\}$;
7. $\quad$ **for** each vertex $v \in Adj[u]$
8. $\qquad$ Relax(u, v, w);

- Idea:
  - search all shortest paths
    - In a smart way (use dynamic-programming, see next lecture)
  - Then choose a shortest path

Example: Dijkstra's Shortest-Path Algorithm

- Find the shortest path from vertex s to vertex v
  - s→x→u→v ; Weight = 5+3+1



Runtime Analysis of Dijkstra's Algorithm

```
Dijkstra(G, w, s)
1. Initialize-Single-Source(G, s);
2. S ← ∅ ;
3. Q ← V[G];
4. while Q ≠ ∅
5.    u ← Extract-Minimum-Element(Q);
6.    S ← S ∪ {u};
7.    for each vertex v ∈ Adj[u]
8.        Relax(u, v, w);
```

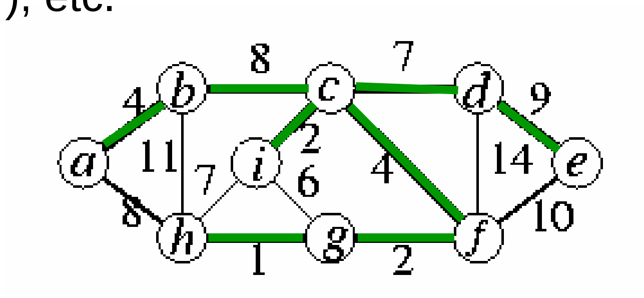
- *Q* is implemented as a linear array: $O(V^2)$.
  - Line 5: $O(V)$ for Extract-Minimum-Element, so $O(V^2)$ with the **while** loop.
  - Lines 7--8: $O(E)$ operations, each takes $O(1)$ time.
- *Q* is implemented as a binary heap: $O(E \lg V)$.
- *Q* is implemented as a Fibonacci heap: $O(E + V \lg V)$.

Dijkstra's SP Pseudo Code in Text

```
struct vertex {
  ...
  int distance;
};

dijkstra(set of struct vertex V, struct vertex v_s, struct vertex v_t)
{
  set of struct vertex T;
  struct vertex u, v;
  V ← V \ {v_s};
  T ← {v_s};
  v_s.distance ← 0;
  for each u ∈ V
    if ((v_s, u) ∈ E)
      u.distance ← w((v_s, u))
    else u.distance ← +∞;
  while (v_t ∉ T) {
     u ← "u ∈ V, such that ∀v ∈ V : u.distance ≤ v.distance";
     T ← T ∪ {u};
     V ← V \ {u};
     for each v "such that (u, v) ∈ E"
       if (v.distance > w((u, v)) + u.distance)
         v.distance ← w((u, v)) + u.distance;
  }
}
```

Minimum Spanning Tree (MST)

- Given an undirected graph $G = (V, E)$ with weights on the edges, a **minimum spanning tree (MST)** of G is a subset $T \subseteq E$ such that
  - T has no cycles
  - T contains all vertices in V
  - sum of the weights of all edges in T is minimum.
- Number of edges in T is number of vertices minus one
- Applications: circuit interconnection (minimizing tree **radius**), communication network (minimizing tree **diameter**), etc.



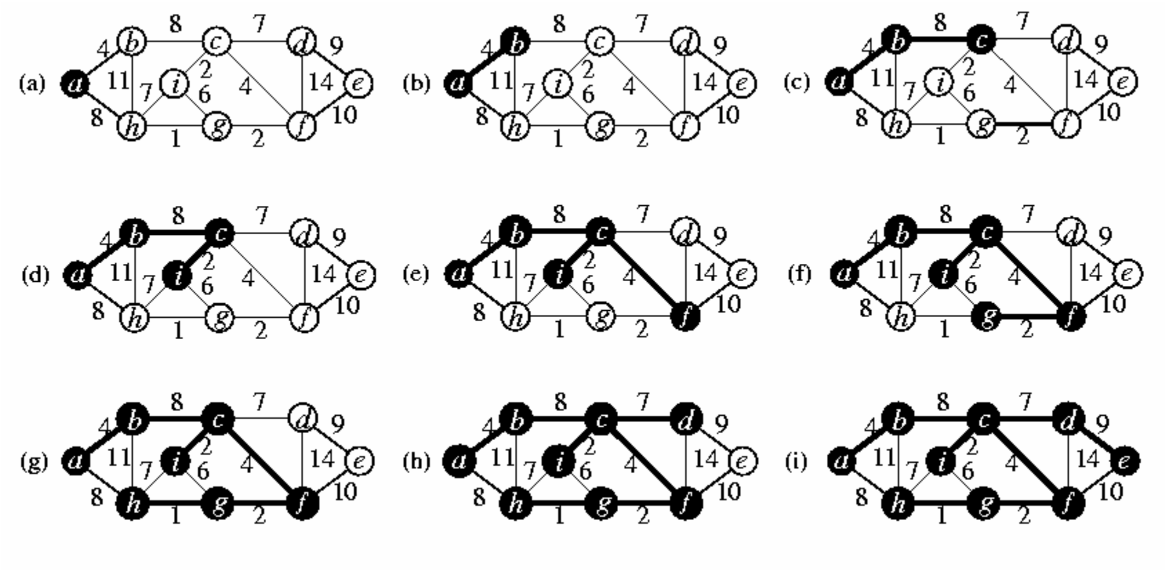
Prim's MST Algorithm

```
MST-Prim(G,w,r)
Q ← V[G];
for  each vertex u ∈ Q
   key[u] ← ∞;
key[r] ← 0;
π[r] ←  NIL;
while  Q ≠ ∅
   u ← Extract-Minimum-Element(Q);
   for  each vertex v ∈ Adj[u]
      if v ∈ Q and w(u,v) < key[v]
         π[v] ← u;
         key[v] ← w(u,v)
```

- *Q*
  priority queue for vertices not in the tree, based on key[].
- *Key[]*
  min weight of any edge connecting to a vertex in the tree.

- Starts from a vertex and grows until the **tree** spans all the vertices.
  - The edges in *A* always form a single tree.
  - At each step, a safe, minimum-weighted edge connecting a vertex in *A* to a vertex in *V* - *A* is added to the tree.

Example: Prim's MST Algorithm



Time Complexity of Prim's MST Algorithm

```
MST-Prim(G,w, r)
1. Q ← V[G];
2. for  each vertex u ∈ Q
3.    key[u] ← ∞;
4. key[r] ← 0;
5. π[r] ←  NIL;
6. while  Q ≠ ∅
7.    u ← Extract-Minimum-Element(Q);
8.    for  each vertex v ∈ Adj[u]
9.       if v ∈ Q and w(u,v) < key[v]
10.           πv] ← u;
11.           key[v] ← w(u,v)
```

- Straightforward implementation: O(V^2) time
  - Lines 1--5: *O*(*V*).
  - Line 7: *O*(*V*) for Extract-Minimum-Element, so *O*(V^2) with the **while** loop.
  - Lines 8--11: *O*(*E*) operations, each takes *O*(lg*V*) time.
- Run in O(E lg V) time if *Q* is implemented as a binary heap
- Run in O(E + VlgV) time if *Q* is implemented as a Fibonacci heap

Prim's MST Pseudo Code in Text

```
prim(set of struct vertex V)
{
  set of struct edge F;
  set of struct vertex W;
  struct vertex u;
  u ← "any vertex from V";
  V ← V \ {u};
  W ← {u};
  F ← Ø;
  for each v ∈ V
    if ((u, v) ∈ E) {
      v.distance ← w((u, v));
      v.via_edge ← (u, v);
    }
    else v.distance ← +∞;
  while (V ≠ Ø) {
     u ← "u ∈ V, such that ∀v ∈ V : u.distance ≤ v.distance";
     W ← W ∪ {u};
     V ← V \ {u};
     F ← F ∪ {u.via_edge};
     for each v "such that (u, v) ∈ E"
       if (v.distance > w((u, v))) {
         v.distance ← w((u, v));
         v.via_edge ← (u, v);
       }
  }
}
```