Module #4.5, Topic # □: Cardinality & Infinite Sets

Rosen 5th ed., last part of §3.2
~10 slides, ½ lecture
Module #4 - Functions

Infinite Cardinalities (from §3.2)

- Using what we learned about functions in §1.8, it’s possible to formally define cardinality for infinite sets.
- We show that infinite sets come in different sizes of infinite!
- This also gives us some interesting proof examples.

Cardinality: Formal Definition

- For any two (possibly infinite) sets $A$ and $B$, we say that $A$ and $B$ have the same cardinality (written $|A|=|B|$) iff there exists a bijection (bijective function) from $A$ to $B$.
- When $A$ and $B$ are finite, it is easy to see that such a function exists iff $A$ and $B$ have the same number of elements $n\in\mathbb{N}$. 
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Countable versus Uncountable

- For any set $S$, if $S$ is finite or if $|S| = |\mathbb{N}|$, we say $S$ is countable. Else, $S$ is uncountable.
- Intuition behind “countable;” we can enumerate (generate in series) elements of $S$ in such a way that any individual element of $S$ will eventually be counted in the enumeration. Examples: $\mathbb{N}$, $\mathbb{Z}$.
- Uncountable: No series of elements of $S$ (even an infinite series) can include all of $S$’s elements. Examples: $\mathbb{R}$, $\mathbb{R}^2$, $\mathcal{P}(\mathbb{N})$.

Countable Sets: Examples

- **Theorem:** The set $\mathbb{Z}$ is countable.
  - **Proof:** Consider $f: \mathbb{Z} \rightarrow \mathbb{N}$ where $f(i) = 2i$ for $i \geq 0$ and $f(i) = -2i - 1$ for $i < 0$. Note $f$ is bijective.
- **Theorem:** The set of all ordered pairs of natural numbers $(n, m)$ is countable.
  - Consider listing the pairs in order by their sum $s = n + m$, then by $n$. Every pair appears once in this series; the generating function is bijective.
Uncountable Sets: Example

**Theorem:** The open interval 
\[0,1) \equiv \{ r \in \mathbb{R} \mid 0 \leq r < 1 \} \] is uncountable.

**Proof** by diagonalization: (Cantor, 1891)
- Assume there is a series \( \{r_i\} = r_1, r_2, \ldots \)
  containing all elements \( r \in [0,1) \).
- Consider listing the elements of \( \{r_i\} \) in decimal notation (although any base will do) in order of increasing index: ... (continued on next slide)

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Uncountability of Reals, cont’d

A postulated enumeration of the reals:
\begin{align*}
r_1 &= 0.d_{1,1} d_{1,2} d_{1,3} d_{1,4} d_{1,5} d_{1,6} d_{1,7} d_{1,8} \ldots \\
r_2 &= 0.d_{2,1} d_{2,2} d_{2,3} d_{2,4} d_{2,5} d_{2,6} d_{2,7} d_{2,8} \ldots \\
r_3 &= 0.d_{3,1} d_{3,2} d_{3,3} d_{3,4} d_{3,5} d_{3,6} d_{3,7} d_{3,8} \ldots \\
r_4 &= 0.d_{4,1} d_{4,2} d_{4,3} d_{4,4} d_{4,5} d_{4,6} d_{4,7} d_{4,8} \ldots \\
\end{align*}

- Now, consider a real number generated by taking all digits \( d_{i,i} \) that lie along the diagonal in this figure and replacing them with different digits.

That real doesn't appear in the list!
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Uncountability of Reals, fin.

- E.g., a postulated enumeration of the reals:
  \( r_1 = 0.301948571... \)
  \( r_2 = 0.103918481... \)
  \( r_3 = 0.039194193... \)
  \( r_4 = 0.918237461... \)

- OK, now let’s add 1 to each of the diagonal digits (mod 10), that is changing 9’s to 0.
  - 0.4103… can’t be on the list anywhere!

Transfinite Numbers

- The cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of the natural numbers, \( \aleph_0 \equiv |\mathbb{N}| \), is the *first transfinite cardinal* number. (There are none smaller.)
- The *continuum hypothesis* claims that \( |\mathbb{R}| = \aleph_1 \), the *second transfinite cardinal*.
Do Uncountable Sets Really Exist?

- The set of objects that can be defined using finite-length strings of symbols ("descriptions") is only countable.
- Therefore, any uncountable set must consist primarily of elements which individually have no finite description.
- Löwenheim-Skolem theorem: No consistent theory can ever force an interpretation involving uncountables.
- The "constructivist school" asserts that only objects constructible from finite descriptions exist. \(\neg \exists R\)
- Most mathematicians are happy to use uncountable sets anyway, because postulating their existence has not led to any demonstrated contradictions (so far).

Countable vs. Uncountable

- You should:
  - Know how to define "same cardinality" in the case of infinite sets.
  - Know the definitions of countable and uncountable.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.