Theorem proving

Formal Methods
Lecture 8

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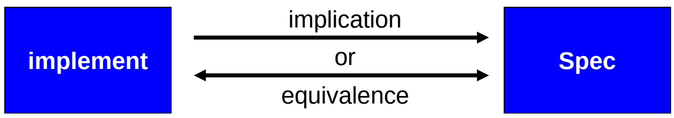
Theorem Proving: Historical Perspective

- Theorem proving (or automated deduction) = logical deduction performed by machine
- At the intersection of several areas
  - Mathematics: original motivation and techniques
  - Logic: the framework and the meta-reasoning techniques

Theorem proving

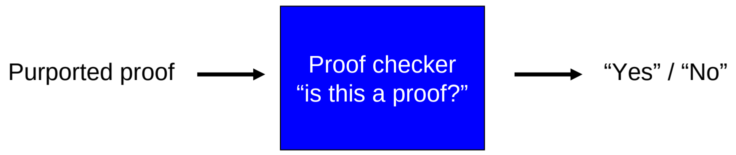
- Prove that an implementation satisfies a specification by mathematical reasoning



Theorem proving

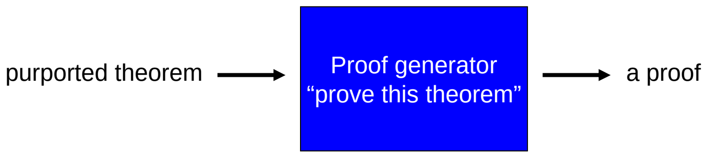
- Implementation and specification expressed as *formulas* in *a formal logic*
- Required relationship (logical equivalence/logical implication) described as *a theorem* to be proven within the context of a proof calculus
- A proof system:
  - A set of axioms and inference rules (simplification, rewriting, induction, etc.)

Proof checking



- It is a purely *syntactic* matter to decide whether each theorem is an axiom or follows from previous theorems (axioms) by a rule of inference

Proof generation



- Complete automation generally impossible: theoretical undecidability limitations
- However, a great deal can be automated (decidable subsets, specific classes of applications and specification styles)

Applications

- Hardware and software verification (or debugging)
- Automatic program synthesis from specifications
- Discovery of proofs of conjectures
  - A conjecture of Tarskiwas proved by machine (1996)
  - There are effective geometry theorem provers

Program Verification

- Fact: mechanical verification of software would improve software productivity, reliability, efficiency
- Fact: such systems are still in experimental stage
  - After 40 years !
  - Research has revealed formidable obstacles
  - Many believe that program verification is extremely difficult

Program Verification

- Fact:
  - Verification is done with respect to a specification
  - Is the specification simpler than the program ?
  - What if the specification is not right ?
- Answer:
  - Developing specifications is hard
  - Still redundancy exposes many bugs as inconsistencies
  - We are interested in partial specifications
    - An index is within bounds, a lock is released…

Programs, Theorems. Axiomatic Semantics

- Consists of:
  - A language for writing specifications about programs
  - Rules for establishing when specifications hold
- Typical specifications:
  - During the execution, only non-null pointers are dereferenced
  - This program terminates with x = 0
- Partial vs. total correctness specifications
  - Safety vs. liveness properties
  - Usually focus on safety (partial correctness)

Specification Languages

- Must be easy to use and expressive (conflicting needs)
  - Most often only expressive
- Typically they are extensions of first-order logic
  - Although higher-order or modal logics are also used
- We focus here on state-based specifications (safety)
  - State = values of variables + contents of heap (+ past state)
    - Not allowed: "variable x is live", "lock L will be released", "there is no correlation between the values of x and y"

A Specification Language

- We'll use a fragment of first-order logic:
  - Formulas P ::= A | true | false | P1 $\wedge$ P2| P1 $\vee$ P2| ¬P | $\forall$x.P
  - Atoms A ::= E1≤E2| E1= E2| f(A1,…,An) | …
  - ※ All boolean expressions from our language are atoms
- Can have an arbitrary collection of predicate symbols
  - reachable(E1,E2) - list cell E2 is reachable from E1
  - sorted(a, L, H) - array a is sorted between L and H
  - ptr(E,T) - expression E denotes a pointer to T
  - E : ptr(T) - same in a different notation
- An assertion can hold or not in a given state
  - Equivalently, an assertion denotes a set of states

Program Verification Using Hoare′s Logic

Hoare Triples

- Partial correctness: { P } s { Q }
  - When you start s in any state that satisfies P
  - If the execution of s terminates
  - It does so in a state that satisfies Q
- Total correctness: [P] s [Q]
  - When you start sin any state that satisfies P
  - The execution of s terminates and
  - It does so in a state that satisfies Q
- Defined inductively on the structure of statements

Hoare Rules

- Assignments
  - y:=t
- Composition
  - S1; S2
- If-then-else
  - if e the S1 else S2
- While
  - while e do S
- Consequence

Greatest common divisor

```
{x1>0 ∧ x2>0}
y1:=x1;
y2:=x2;
while ¬(y1=y2) do
    if y1>y2 then y1:=y1-y2
                 else y2:=y2-y1
{y1=gcd(x1,x2)}
```

Why it works?

- Suppose that y1,y2 are both positive integers.
  - If y1>y2 then gcd(y1,y2)=gcd(y1-y2,y2)
  - If y2>y1 then gcd(y1,y2)=gcd(y1,y2-y1)
  - If y1-y2 then gcd(y1,y2)=y1=y2

Hoare Rules: Assignment

- General rule:
  - {p[t/y]} y:=t {p}
- Examples:
  - {y+5=10} y:=y+5 {y=10}
  - {y+y<z} x:=y {x+y<z}
  - {2*(y+5)>20} y:=2*(y+5) {y>20}
- Justification: write p with y’ instead of y, and add the conjunct y’=t. Next, eliminate y’ by replacing y’ by t.

Hoare Rules: Assignment

{p} y:=t {?}

- Strategy: write p and the conjunct y=t, where y' replaces y in both p and t. Eliminate y'.

Example:

{y>5} y:=2*(y+5) {?}

{p} y:=t {∃y' (p[y'/y] ∧ t[y'/y]=y)}

y'>5 ∧ y=2*(y'+5) → y>20

Hoare Rules: Composition

- General rule:
  - {p} S1 {r}, {r} S2 {q} → {p} S1;S2 {q}
- Example:

  if the antecedents are

  1. {x+1=y+2} x:=x+1 {x=y+2}
  2. {x=y+2} y:=y+2 {x=y}

  Then the consequent is

  {x+1=y+2} x:=x+1; y:=y+2 {x=y}

Hoare Rules: If-then-else

- General rule:
  - {p ∧ e} S1 {q}, {p ∧ ¬e} S2 {q}
  - {p} if e then S1 else S2 {q}
- Example:

  p is gcd(y1,y2)=gcd(x1,x2) ∧ y1>0 ∧ y2>0 ∧ ¬(y1=y2)

  e is y1>y2

  S1 is y1:=y1-y2

  S2 is y2:=y2-y1

  q is gcd(y1,y2)=gcd(x1,x2) ∧ y1>0 ∧ y2>0

Hoare Rules: While

- General rule:
  - {p ∧ e} S {p}
  - {p} while e do S {p ∧ ¬e}
- Example:

  p is {gcd(y1,y2)=gcd(x1,x2) ∧ y1>0 ∧ y2>0}

  e is (y1 ≠ y2)

  S is if y1>y2 then y1:=y1-y2 else y2:= y2-y1

Hoare Rules: Consequence

- Strengthen a precondition
  - r→p, {p} S {q}
  - {r} S {q}
- Weaken a postcondition
  - {p} S {q}, q→r
  - {p} S {r}

Soundness

- Hoare logic is sound in the sense that everything that can be proved is correct!
- This follows from the fact that each axiom and proof rule preserves soundness.

Completeness

- A proof system is called complete if every correct assertion can be proved.

- Propositional logic is complete.
- No deductive system for the standard arithmetic can be complete (Godel).

And for Hoare logic?

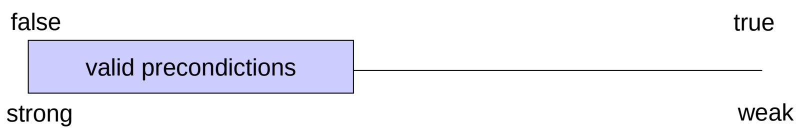
- Let S be a program and p its precondition.
- Then {p} S {false} means that S never terminates when started from p. This is undecidable. Thus, Hoare's logic cannot be complete.

Hoare Rules: Examples

- Consider
  - $\{ x = 2 \}\ x := x + 1\ \{ x < 5 \}$
  - $\{ x < 2 \}\ x := x + 1\ \{ x < 5 \}$
  - $\{ x < 4 \}\ x := x + 1\ \{ x < 5 \}$
- They all have correct preconditions
- But the last one is the most general (or weakest) precondition

Dijkstra's Weakest Preconditions

- Consider $\{ P \}\ s\ \{ Q \}$
- Predicates form a lattice:



- To verify $\{ P \}\ s\ \{ Q \}$
- compute $WP(s, Q)$ and prove $P \neq WP(s, Q)$

Weakest prendition, Strongest postcondition

- For an assertion p and code S, let post(p,S) be the strongest assertion such that
  {p}S{post(p,S)}
  That is, if {p}S{q} then post(p,S)→q.

- For an assertion q and code S, let pre(S,q) be the weakest assertion such that
  {pre(S,q)}S{q}
  That is, if {p}S{q} then p→pre(S,q).

Relative completeness

- Suppose that either
  - post(p,S) exists for each p, S, or
  - pre(S,q) exists for each S, q.
- Some oracle decides on pure implications.

Then each correct Hoare triple can be proved.
What does that mean? The weakness of the proof system stem from the weakness of the (FO) logic, not of Hoare's proof system.

Extensions

Many extensions for Hoare's proof rules:

- Total correctness
- Arrays
- Subroutines
- Concurrent programs
- Fairness

Higher-Order Logic

Higher-Order Logic

- *First-order logic*:
  - only domain variables can be quantified.
- *Second-order logic*:
  - quantification over subsets of variables (i.e., over predicates).
- *Higher-order logics*:
  - quantification over arbitrary predicates and functions.

Higher-Order Logic

- Variables can be functions and predicates,
- Functions and predicates can take functions as arguments and return functions as *values,*
- Quantification over functions and predicates.

- Since arguments and results of predicates and functions can themselves be predicates or functions, this imparts a **first-class status** to functions, and allows them to be manipulated just like *ordinary values*

Higher-Order Logic

- **Example 1:** (mathematical induction)
  $\forall P. [P(0) \wedge (\forall n. P(n) \rightarrow P(n+1))] \rightarrow \forall n.P(n)$
  (Impossible to express it in FOL)

- **Example 2:**
  Function Rise defined as $Rise(c, t) = \neg c(t) \wedge c(t+1)$
  Rise expresses the notion that a signal *c* rises at time *t*.
  Signal is modeled by a function $c: N \rightarrow \{F,T\}$, passed as argument to Rise.
  Result of applying Rise to c is a function: $N \rightarrow \{F,T\}$.

Higher-Order Logic (cont'd)

- **Advantage:**
  - high expressive power!
- **Disadvantages:**
  - Incompleteness of a sound proof system for most higher-order logics
  - **Theorem** (Gödel, 1931)
    There is no complete deduction system for the second-order logic.
  - Reasoning more difficult than in FOL, need ingenious inference rules and heuristics.

Higher-Order Logic (cont'd)

- **Disadvantages:**
  - Inconsistencies can arise in higher-order systems if semantics not carefully defined

    "*Russell Paradox*":

    Let P be defined by $P(Q) = \neg Q(Q)$. By substituting P for Q, leads to $P(P) = \neg P(P)$,

    Contradiction! (P: bool $\rightarrow$ bool, Q: bool $\rightarrow$ bool)
  - Introduction of "types" (syntactical mechanism) is effective against certain inconsistencies.
  - Use *controlled form of logic and inferences* to minimize the risk of inconsistencies, while gaining the benefits of powerful representation mechanism.
  - Higher-order logic increasingly popular for hardware verification!

Theorem Proving Systems

- Automated deduction systems (e.g. Prolog)
  - full automatic, but only for a decidable subset of FOL
  - speed emphasized over versatility
  - often implemented by ad hoc decision procedures
  - often developed in the context of AI research
- Interactive theorem proving systems
  - semi-automatic, but not restricted to a decidable subset
  - versatility emphasized over speed
  - in principle, a complete proof can be generated for every theorem

Theorem Proving Systems

- **Some theorem proving systems:**
  - Boyer-Moore (first-order logic)
  - HOL (higher-order logic)
  - PVS (higher-order logic)
  - Lambda (higher-order logic)

HOL

- HOL (Higher-Order Logic) developed at University of Cambridge
- Interactive environment (in ML, Meta Language) for machine assisted theorem proving in higher-order logic (a proof assistant)
- Steps of a proof are implemented by applying inference rules chosen by the user; HOL checks that the steps are safe
- All inferences rules are built on top of eight primitive inference rules

HOL

- Mechanism to carry out backward proofs by applying built-in ML functions called *tactics* and *tacticals*
- By building complex tactics, the user can customize proof strategies
- Numerous applications in software and hardware verification
- Large user community

HOL Theorem Prover

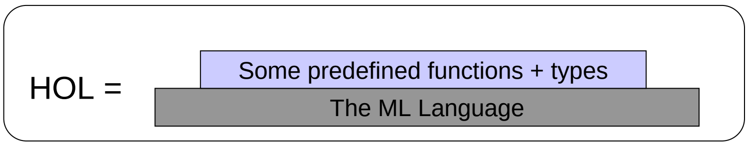
- Logic is strongly typed (type inference, abstract data types, polymorphic types, etc.)
- It is sufficient for expressing most ordinary mathematical theories (the power of this logic is similar to set theory)
- HOL provides considerable built-in theorem-proving infrastructure:
  - a powerful *rewriting* subsystems
  - *library* facility containing useful theories and tools for general use
  - *Decision procedures* for tautologies and semi-decision procedure for linear arithmetic provided as libraries

HOL Theorem Prover

- The primary interface to HOL is the functional programming language ML
- Theorem proving tools are functions in ML (users of HOL build their own application specific theorem proving infrastructure by writing programs in ML)
- Many versions of HOL:
  - HOL88: Classic ML (from LCF);
  - HOL90: Standard ML
  - HOL98: Moscow ML

HOL Theorem Prover (cont'd)

- HOL and ML

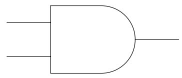


- The HOL systems can be used in two main ways:
  - for directly proving theorems: when higher-order logic is a suitable specification language (e.g., for hardware verification and classical mathematics)
  - as embedded theorem proving support for application-specific verification systems when specification in specific formalisms needed to be supported using customized tools.
- The approach to mechanizing formal proof used in HOL is due to Robin Milner.

  He designed a system, called LCF: Logic for Computable Functions. (The HOL system is a direct descendant of LCF.)

Specification in HOL

- **Functional description**:
  express output signal as function of input signals, e.g.:
  AND gate:
  out = **and** (in1, in2) = (in1 ∧ in2)



- **Relational (predicate) description**:
  gives relationship between inputs and outputs in the form of a predicate (a Boolean function returning "true" of "false"), e.g.:
  AND gate:
  AND ((in1, in2),(out)):= out =(in1 ∧ in2)

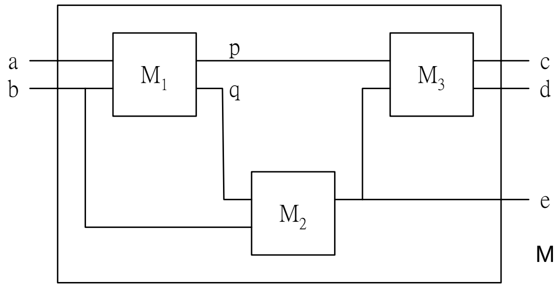
Specification in HOL

- **Notes:**
  - functional descriptions allow recursive functions to be described. They cannot describe bi-directional signal behavior or functions with multiple feed-back signals, though
  - relational descriptions make no difference between inputs and outputs
  - Specification in HOL will be a combination of predicates, functions and abstract types

in_1
in_2

Specification in HOL

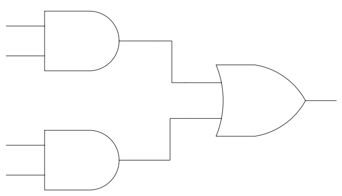
Network of modules



- conjunction "∧" of implementation module predicates
  M (a, b, c, d, e):= M1 (a, b, p, q) ∧ M2 (q, b, e) ∧ M3 (e, p, c, d)
- hide internal lines (p,q) using **existential quantification**
  M (a, b, c, d, e):= ∃ p q. M1 (a, b, p, q) ∧ M2 (q, b, e) ∧ M3 (e, p, c, d)

Specification in HOL

Combinational circuits



SPEC (in1, in2, in3, in4, out):=
 out = (in1 ∧ in2) ∨ (in3 ∧ in4)

IMPL (in1, in2, in3, in4, out):=
 ∃ l1, l2. **AND** (in1, in2, l1) ∧ **AND** (in3, in4, l2) ∧ **OR** (l1, l2, out)
  where **AND** (a, b, c):= (c =a ∧ b)
   OR (a, b, c):= (c = a ∨ b)

Specification in HOL

- **Note**: a functional description would be:
  IMPL (in1, in2, in3, in4, out):=
  out = (**or** (**and** (in1, in2), **and** (in3, in4))
  where **and** (in1, in2) = (in1 $\wedge$ in2)
  or (in1, in2) = (in1 $\vee$ in2)

Specification in HOL

Sequential circuits

- Explicit expression of time (discrete time modeled as natural numbers).
- Signals defined as functions over time, e.g. type: (**nat** $\rightarrow$ **bool**) or (**nat** $\rightarrow$ **bitvec**)
- Example: D-flip-flop (latch):
  DFF (in, out):= (out (0) = F) $\wedge$ ($\forall$ t. out (t+1) = in (t))
  in and *out* are functions of time *t* to boolean values: type (nat $\rightarrow$ bool)

Specification in HOL

- Notion of time can be added to combinational circuits, e.g., AND gate
  AND (in1, in2, out):= $\forall$ t. out (t) = (in1(t) $\wedge$ in2(t))

- Temporal operators can be defines as predicates, e.g.:
  EVENTUAL sig t1 = $\exists$ t2. (t2 > t1) $\wedge$ sig t2
  meaning that signal "sig" will eventually be true at time t2 > t1 .

- Note: This kind of specification using existential quantified time variables is useful to describe asynchronous behavior

HOL Proof Mechanism

- A formal proof is a sequence, each of whose elements is
  - either an *axiom*
  - or follows from earlier members of the sequence by a *rule of inference*
- A *theorem* is the last element of a proof
- A *sequent* is written:
  - $\Gamma \vdash P$, where Γ is a *set of assumptions* and P is the *conclusion*

HOL Proof Mechanism

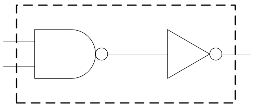
- In HOL, this consists in applying ML functions representing rules of inference to axioms or previously generated theorems
- The sequence of such applications directly correspond to a proof
- A value of *type* **thm** can be obtained either
  - directly (as an axiom)
  - by computation (using the built-in functions that represent the inference rules)
- ML typechecking ensures these are the only ways to generate a thm:

  All theorems must be proved!

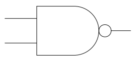
Verification Methodology in HOL

1. Establish a formal specification (predicate) of the intended behavior (SPEC)
2. Establish a formal description (predicate) of the implementation (IMP), including:
   - behavioral specification of all sub-modules
   - structural description of the network of sub-modules
3. Formulation of a proof goal, either
   - IMP $\Rightarrow$ SPEC (proof of implication), or
   - IMP $\Leftrightarrow$ SPEC (proof of equivalence)
4. Formal verification of above goal using a set of inference rules

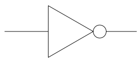
Example 1: Logic AND



- **AND Specification:**
  - AND_SPEC (i1,i2,out) := out = i1 ∧ i2
- **NAND specification:**
  - NAND (i1,i2,out) := out = ¬(i1 ∧ i2)



- **NOT specification:**
  - NOT (i, out) := out = ¬ I



- **AND Implementation:**
  - AND_IMPL (i1,i2,out) := ∃x. NAND (i1,i2,x) ∧ NOT (x,out)

i_1
i_2

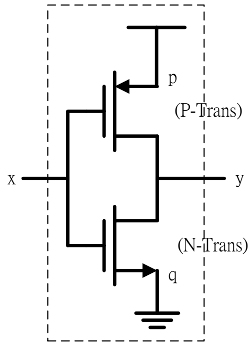
Example 1: Logic AND

- **Proof Goal:**
  - ∀ i1, i2, out. AND_IMPL(i1,i2,out) ⇒ ANDSPEC(i1,i2,out)
- **Proof (forward)**

AND_IMP(i1,i2,out) {from above circuit diagram}
⊢ ∃ x. NAND (i1,i2,x) ∧ NOT (x,out) {by def. of AND impl}
⊢ NAND (i1,i2,***x***) ∧ NOT(***x***,out) {strip off "∃ x."}
⊢ NAND (i1,i2,***x***) {left conjunct of line 3}
⊢ x =¬ (i1 ∧ i2) {by def. of NAND}
⊢ NOT (*x*,out) {right conjunct of line 3}
⊢ out = ¬ x {by def. of NOT}
⊢ out = ¬(¬(i1 ∧ i2) {substitution, line 5 into 7}
⊢ out =(i1 ∧ i2) {simplify, ¬¬ t=t}
⊢ AND (i1,i2,out) {by def. of AND spec}
⊢ AND_IMPL (i1,i2,out) ⇒ AND_SPEC (i1,i2,out)
Q.E.D.

Example 2: CMOS-Inverter

- **Specification** (black-box behavior)
  - Spec(x,y):= (y = ¬ x)
- **Implementation**



- **Basic Modules Specs**
  - PWR(x):= (x = T)
  - GND(x):= (x = F)
  - N-Trans(g,x,y):= (g ⇒ (x = y))
  - P-Trans(g,x,y):= (¬ g ⇒ (x = y))

Example 2: CMOS-Inverter

- **Implementation (network structure)**
  - Impl(x,y):= ∃ p, q.
    PWR(p) ∧
    GND(q) ∧
    N-Tran(x,y,q) ∧
    P-Tran(x,p,y)
- **Proof goal**
  - ∀ x, y. Impl(x,y) ⇔ Spec(x,y)
- **Proof (forward)**
  - Impl(x,y):= ∃ p, q.
    (p = T) ∧
    (q = F) ∧ (substitution of the definition of PWR and GND)
    N-Tran(x,y,q) ∧
    P-Tran(x,p,y)

Example 2: CMOS-Inverter

- Impl(x,y):= ∃ p q.
  (p = T) ∧
  (q = F) ∧ (substitution of p and q in P-Tran and N-Tran)
  N-Tran(x,y,F) ∧
  P-Tran(x,T,y)
- Impl(x,y):=
  (∃ p. p = T) ∧
  (∃ q. q = F) ∧ (use Thm: "∃a. t1 ∧ t2 = (∃a. t1) ∧ t2" if a is free in t2)
  N-Tran(x,y,F) ∧
  P-Tran(x,T,y)

Example 2: CMOS-Inverter

- Impl(x,y):=
  T ∧
  T ∧ (use Thm: "(∃a. a=T) = T" and "(∃a. a=F) = T")
  N-Tran(x,y,F) ∧
  P-Tran(x,T,y)
- Impl(x,y):=
  N-Tran(x,y,F) ∧ (use Thm: "x ∧ T = x")
  P-Tran(x,T,y)
- Impl(x,y):=
  (x ⇒ (y = F)) ∧ (use def. of N-Tran and P-Tran)
  (¬ x ⇒ (T = y))

Example 2: CMOS-Inverter

- Impl(x,y):=
  $(\neg x \vee (y = F)) \wedge$ ((use "$(a \Rightarrow b) = (\neg a \vee b)$")
  $(x \vee (T = y))$

- Boolean simplifications:
  - Impl(x,y):= $(\neg x \wedge x) \vee (\neg x \wedge (T = y)) \vee ((y = F) \wedge x) \vee ((y = F) \wedge (T = y))$
  - Impl(x,y):= $F \vee (\neg x \wedge (T = y)) \vee ((y = F) \wedge x) \vee F$
  - Impl(x,y):= $(\neg x \wedge (T = y)) \vee ((y = F) \wedge x)$

Example 2: CMOS-Inverter

- Case analysis x=T/F
  - x=T:Impl(T,y):= $(F \wedge (T = y)) \vee ((y = F) \wedge T)$
  - x=F:Impl(F,y):= $(T \wedge (T = y)) \vee ((y = F) \wedge F)$

  - x=T:Impl(T,y):= (y = F)
  - x=F:Impl(F,y):= (T = y)

  =

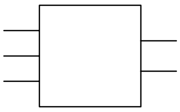
- Case analysis on **Spec**:
  - x=T:Spec(T,y):= (y = F)
  - x=F:Spec(F,y):= (y = T)

- **Conclusion**: $\vdash$Spec(x,y) $\Leftrightarrow$ Impl(x,y)

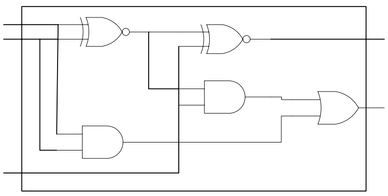
Abstraction Forms

- **Structural abstraction:**
  - only the behavior of the external inputs and outputs of a module is of interest (abstracts away any internal details)
- **Behavioral abstraction:**
  - only a specific part of the total behavior (or behavior under specific environment) is of interest
- **Data abstraction:**
  - behavior described using abstract data types (e.g. natural numbers instead of Boolean vectors)
- **Temporal abstraction:**
  - behavior described using different time granularities (e.g. refinement of instruction cycles to clock cycles)

Example 3: 1-bit Adder



- **Specification:**
  - ADDER_SPEC (in1:**nat**, in2:**nat**, cin:**nat**, sum:**nat**, cout:**nat**):= in1+in2 + cin = 2*cout + sum
- **Implementation**:



- **Note**: Spec is a **structural abstraction** of Impl.

1-bit Adder (cont'd)

- **Implementation**:

  ADDER_IMPL(in1:**bool**, in2:**bool**, cin:**bool**, sum:**bool**, cout:**bool**):=

  $\exists$ l1 l2 l3. EXOR (in1, in2, l1) $\wedge$
  AND (in1, in2, l2) $\wedge$
  EXOR (l1,cin,sum) $\wedge$
  AND (l1, cin, l3) $\wedge$
  OR (l2, l3, cout)

- Define a **data abstraction function** (**bn: bool** $\rightarrow$ **nat**) needed to relate Spec variable types (nat) to Impl variable types (bool):

$$bn(x) := \begin{cases} 1, \text{ if } x = T \\ 0, \text{ if } x = F \end{cases}$$

1-bit Adder (cont'd)

- **Proof goal**:

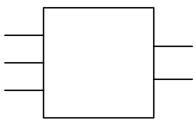
  $\forall$ in1, in2, cin, sum, cout.
  ADDER_IMPL (in1, in2, cin, sum, cout)
  $\Rightarrow$ ADDER_SPEC (**bn**(in1), **bn**(in2), **bn**(cin), **bn**(sum), **bn**(cout))

Verification of Generic Circuits

- used in datapath design and verification
- **idea**:
  - verify **n**-bit circuit then specialize proof for specific value of **n**, (i.e., once proven for **n**, a simple instantiation of the theorem for any concrete value, e.g. 32, gets a proven theorem for that instance).
- use of induction proof

Example 4: N-bit Adder

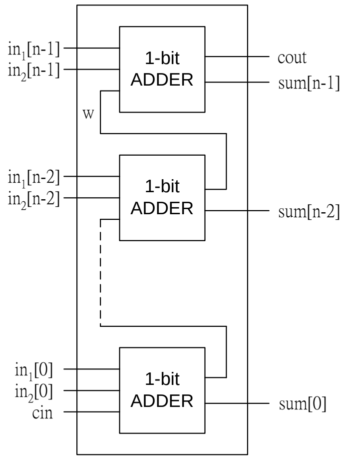
- **N-bit Adder**



- **Specification**
  - N-ADDER_SPEC (**n**,in1,in2,cin,sum,cout):= (in1 + in2 + cin = 2^{n+1} * cout + sum)

Example 4: N-bit Adder

- **Implementation**



N-bit Adder (cont'd)

Implementation

- recursive definition:

  N-ADDER_IMP(n,in1[0..n-1],in2[0..n-1],cin,sum[0..n-1],cout):=
  ∃ w. N-ADDER_IMP(n-1,in1[0..n-2],in2[0..n-2],cin,sum[0..n-2],w) ∧
  N-ADDER_IMP(1,in1[n-1],in2[n-1],w,sum[n-1],cout)
- Note:
  - N-ADDER_IMP(1,in1[i],in2[i],cin,sum[i],cout) = ADDER_IMP(in1[i],in2[i],cin,sum[i],cout)
- Data abstraction function (**vn: bitvec** → **nat**) to relate bit vctors to natural numbers:
  - vn(x[0]):= bn(x[0])
  - vn(x[0,n]):= 2^n * bn(x[n]) + vn(x[0,n-1]

N-bit Adder (cont'd)

- **Proof goal:**

  ∀ **n**, in1, in2, cin, sum, cout.

  N-ADDER_IMP(n,in1[0..n-1],in2[0..n-1],cin,sum[0..n-1],cout)

  ⇒ N-ADDER_SPEC(n, **vn**(in1[0..n-1]), **vn**(in2[0..n-1]), **vn**(cin), **vn**(sum[0..n-1]), **vn**(cout))

- can be **instantiated** with **n = 32:**

  ∀ in1, in2, cin, sum, cout.

  N-ADDER_IMP(in1[0..31],in2[0..31],cin,sum[0..31],cout)

  ⇒ N-ADDER_SPEC(**vn**(in1[0..31]), **vn**(in2[0..31]), **vn**(cin), **vn**(sum[0..31]), **vn**(cout))

N-bit Adder (cont'd)

Proof by induction over n:

- basis step:

  N-ADDER_IMP(0,in1[0],in2[0],cin,sum[0],cout)

  ⇒ N-ADDER_SPEC(0,**vn**(in1[0]),**vn**(in2[0]),**vn**(cin),**vn**(sum[0]),**vn**(cout))

- induction step:

  [N-ADDER_IMP(n,in1[0..n-1],in2[0..n-1],cin,sum[0..n-1],cout) ⇒ N-ADDER_SPEC(n,**vn**(in1[0..n-1]),**vn**(in2[0..n-1]),**vn**(cin),**vn**(sum[0..n-1]),**vn**(cout))**]**

  ⇒

  [N-ADDER_IMP(n+1,in1[0..n],in2[0..n],cin,sum[0..n],cout) ⇒ N-ADDER_SPEC(n+1,**vn**(in1[0..n]),**vn**(in2[0..n]),**vn**(cin),**vn**(sum[0..n]),**vn**(cout))**]**

N-bit Adder (cont'd)

Notes:

- basis step is equivalent to 1-bit adder proof, i.e.

  ADDER_IMP(in1[0],in2[0],cin,sum[0],cout)

  ⇒ ADDER_SPEC(**bn**(in1[0]),**bn**(in2[0]),**bn**(cin),**bn**(sum[0]),**bn**(cout))
- induction step needs more creativity and work load!

Practical Issues of Theorem Proving

No fully automatic theorem provers. All require human guidance in indirect form, such as:

- When to delete redundant hypotheses, when to keep a copy of a hypothesis
- Why and how (order) to use lemmas, what lemma to use is an art
- How and when to apply rules and rewrites
- Induction hints (also nested induction)

Practical Issues of Theorem Proving

- Selection of proof strategy, orientation of equations, etc.
- Manipulation of quantifiers (forall, exists)
- Instantiation of specification to a certain time and instantiating time to an expression
- Proving lemmas about (modulus) arithmetic
- Trying to prove a false lemma may be long before abandoning

PVS

Prototype Verification System (PVS)

- Provides an integrated environment for the development and analysis of formal specifications.
- Supports a wide range of activities involved in creating, analyzing, modifying, managing, and documenting theories and proofs.

Prototype Verification System (cont')

- The primary purpose of PVS is to provide formal support for conceptualization and debugging in the early stages of the lifecycle of the design of a hardware or software system.
- In these stages, both the requirements and designs are expressed in abstract terms that are not necessarily executable.

Prototype Verification System (cont')

- The primary emphasis in the PVS proof checker is on supporting the construction of readable proofs.
- In order to make proofs easier to develop, the PVS proof checker provides a collection of powerful proof commands to carry out propositional, equality, and arithmetic reasoning with the use of definitions and lemmas.

The PVS Language

- The specification language of PVS is built on higher-order logic
  - Functions can take functions as arguments and return them as values
  - Quantification can be applied to function variables
- There is a rich set of built-in types and type-constructors, as well as a powerful notion of subtype.
- Specifications can be constructed using definitions or axioms, or a mixture of the two.

The PVS Language (cont')

- Specifications are logically organized into parameterized *theories* and *datatypes*.
- Theories are linked by *import* and *export* lists.
- Specifications for many foundational and standard theories are preloaded into PVS as prelude theories that are always available and do not need to be explicitly imported.

A Brief Tour of PVS

- Creating the specification
- Parsing
- Typechecking
- Proving
- Status
- Generating LATEX

A Simple Specification Example

```
sum: Theory
   BEGIN
        n: VAR nat
        sum(n): RECURSIVE nat =
        (IF n = 0 THEN 0  ELSE n + sum(n-1) ENDIF)
        MEASURE (LAMBDA n : n)
        closed_form: THEOREM sum(n) = (n * (n + 1)) / 2
   END sum
```

Creating the Specification

- Create a file with a *.pvs* extension
  - Using the M-x new-pvs-file command (M-x nf) to create a new PVS file, and typing sum when prompted. Then type in the sum specification.
  - Since the file is included on the distribution tape in the Examples/tutorial subdirectory of the main PVS directory, it can be imported with the M-x import-pvs-file command (M-x imf). Use the M-x whereis-pvs command to find the path of the main PVS directory.
  - Finally, any external means of introducing a file with extension .pvs into the current directory will make it available to the system. ex: using vi.

Parsing

- Once the sum specification is displayed, it can be parsed with the *M-x parse (M-x pa)* command, which creates the internal abstract representation for the theory described by the specification.
- If the system finds an error during parsing, an error window will pop up with an error message, and the cursor will be placed in the vicinity of the error.

Typechecking

- To typecheck the file by typing M-x typecheck (M-x tc, C-c t), which checks for semantic errors, such as undeclared names and ambiguous types.
- Typechecking may build new files or internal structures such as TCCs. (when sum has been typechecked, a message is displayed in the minibuffer indicating the two TCCs were generated)

Typechecking (cont')

- These TCCs represent proof obligations that must be discharged before the sum theory can be considered typechecked.
- TCCs can be viewed using the M-x show-tccs command.

Typechecking (cont')

```
% Subtype TCC generated (line 7) for n-1
   % unchecked
sum_TCC1: OBLIGATION (FORALL (n : nat) : NOT n=0 IMPLIES n-1 >= 0);

% Termination  TCC generated (line 7) for sum
   % unchecked
sum_TCC2: OBLIGATION (FORALL (n : nat) : NOT n=0 IMPLIES n-1 < n);
```

Typechecking (cont')

- The first TCC is due to the fact that sum takes an argument of type nat, but the type of the argument in the recursive call to sum is integer, since nat is not closed under substraction.
  - Note that the TCC includes the condition NOT n=0, which holds in the branch of the IF-THEN-ELSE in which the expression n-1 occirs.
- The second TCC is needed to ensure that the function sum is total. PVS does not directly support partial functions, although its powerful subtyping mechanism allows PVS to express many operations that are traditionally regarded as partial.
  - The measure function is used to show that recursive definitions are total by requiring the measure to decrease with each recursive call.

Proving

- We are now ready to try to prove the main theorem
- Place the cursor on the line containing the closed form theorem and type M-x prove M-x pr or C-c p
- A new buer will pop up the formula will be displayed and the cursor will appear at the Rule prompt indicating that the user can interact with the prover

Proving (cont')

- First, notice the display, which consists of a single formula labeled {1} under a dashed line.
- This is a *sequent*: formulas above the dashed lines are called *antecedents* and those below are called *succedents*
  - The interpretation of a sequent is that the conjunction of the antecedents implies the disjunction of the succedents
  - Either or both of the antecedents and succedents may be empty

Proving (cont')

- The basic objective of the proof is to generate a proof tree in which all of the leaves are trivially true
- The nodes of the proof tree are sequents and while in the prover you will always be looking at an unproved leaf of the tree
- The current branch of a proof is the branch leading back to the root from the current sequent
- When a given branch is complete (i.e., ends in a true leaf), the prover automatically moves on to the next unproved branch, or, if there are no more unproven branches, notifies you that the proof is complete

Proving (cont')

- We will prove this formula by induction n.
  - To do this, type (induct "n")
  - This generates two subgoals the one displayed is the base case where n is 0
  - To see the inductive step type (postpone) which postpones the current subgoal and moves on to the next unproved one Type (postpone) a second time to cycle back to the original subgoal (labeled closed_form.1)

Proving (cont')

- To prove the base case, we need to expand the denition of sum, which is done by typing (expand "sum")
- After expanding the denition of sum, we send the proof to the PVS decision procedures, which automatically decide certain fragments of arithmetic, by typing (assert)
- This completes the proof of this subgoal and the system moves on to the next subgoal which is the inductive step

Proving (cont')

- The first thing to do here is to eliminate the FORALL quantifier
- This can most easily be done with the skolem! command, which provides new constants for the bound variables
- To invoke this command type (skolem!) at the prompt
- The resulting formula may be simplified by typing (flatten), which will break up the succedent into a new antecedent and succedent

Proving (cont')

- The obvious thing to do now is to expand the denition of sum in the succedent. This again is done with the expand command, but this time we want to control where it is expanded, as expanding it in the antecedent will not help.
- So we type (expand "sum" +), indicating that we want to expand sum in the succedent

Proving (cont')

- The final step is to send the proof to the PVS decision procedures by typing (assert)
- The proof is now complete the system may ask whether to save the new proof and whether to display a brief printout of the proof

```
closed_form :

      |-------
{1}        (FORALL (n : nat) : sum(n) = (n * (n + 1)) / 2)

Rule?     (induct "n")
Inducting on n,
this yields 2 subgoals
closed_form.1 :

      |-------
{1}        sum(0) = (0 * (0 + 1)) / 2

Rule?     (postpone)
Postponing closed_form.1
```

```
closed_form.2 :

    |-------
{1}      (FORALL (j : nat) :
               sum(j) = (j * (j + 1)) / 2
                    IMPLIES sum(j + 1) = ((j + 1) * (j + 1 + 1)) / 2

Rule?    (postpone)
Postponing closed_form.2

closed_form.1 :

    |-------
{1}      sum(0) = (0 * (0 + 1)) / 2

Rule?    (expand “sum”)
(IF 0 = 0 THEN 0 ELSE 0 + sum(0 - 1) ENDIF)
```

```
simplifies to 0
Expanding the definition of sum,
this simplifies to:
closed_form.1 :

    |-------
{1}      0 = 0 / 2

Rule?    (assert)
Simplifying, rewriting, and recording with decision procedures,

This completes the proof of closed_form.1.

closed_form.2 :

    |-------
{1}      (FORALL (j : nat) :
               sum(j) = (j * (j + 1)) / 2
                    IMPLIES sum(j + 1) = ((j + 1) * (j + 1 + 1)) / 2
```

Rule? (skolem!)
Skolemizing,
this simplifies to:
closed_form.2 |- - - - - - -
{1} sum(j ! 1) = (j ! 1 * (j ! 1 + 1)) / 2
IMPLIES sum(j ! 1 + 1) = ((j ! 1 + 1) * (j ! 1 + 1 + 1)) / 2

Rule? (flatten)
Applying disjunctive simplification to flatten sequent,
This simplifies to:
closed_form.2 :

{-1}sum(j ! 1) = (j ! 1 * (j ! 1 + 1)) / 2
|- - - - - - -
{1} sum(j ! 1 + 1) = ((j ! 1 + 1) * (j ! 1 + 1 + 1)) / 2

Rule? (expand "sum" +)
(IF j ! 1 + 1 = 0 THEN 0 ELSE j ! 1 + 1 + sum(j ! 1 + 1 - 1) ENDIF)
simplifies to 1 + sum(j ! 1) + j ! 1
Expanding the definition of sum,
this simplifies to:
closed_form.2:

[-1] sum(j ! 1) = (j ! 1 * (j ! 1 + 1)) / 2
|- - - - - - -
{1} 1 + sum(j ! 1) + j ! 1 = (2 + j ! 1 + (j ! 1 * j ! 1 + 2 * j ! 1)) / 2

Rule? (assert)
Simplifying, rewriting, and recording with decision procedures,

This completes the proof of closed_form.2.

Q.E.D

Run time = 5.62 secs.
Real time= 58.95 secs.

Status

- Type M-x status-proof-theory (M-x spt) and you will see a buffer which displays the formulas in sum (including the TCCs), along with an indication of their proof status
  - This command is useful to see which formulas and TCCs still require proofs
- Another useful command is M-x status-proofchain (M-x spc), which analyzes a given proof to determine its dependencies

Generating LATEX

- Type M-x latex-theory-view (M-x ltv). You will be prompted for the theory name — type sum, or just Return if sum is the default
- After a few moments the previewer will pop up displaying the sum theory, as shown below.

Generating LATEX (cont’)

```
sum:     THEORY
   BEGIN
   n: VAR nat
   sum(n): RECURSIVE nat =
        (IF n = 0 THEN 0 ELSE n + sum(n - 1) ENDIF)
         MEASURE ( λ  n : n)
   closed_form: THEOREM sum(n) = (n * (n + 1)) / 2
END sum
```

Generating LATEX (cont’)

- Finally using the M-x latex-proof command, it is possible to generate a LATEX file from a proof

Expanding the definition of sum

closeed_form.2:

$$\{-1\} \quad \sum_{i=0}^{j'} i = (j' \times (j'+1))/2$$

$$\{1\} \quad (\text{IF } j'+1 = 0 \text{ THEN } 0 \text{ ELSE } j'+1+\sum_{i=0}^{j'+1-1} i \text{ ENDIF}) = ((j'+1) \times (j'+1+1))/2$$

Conclusions

Advantages of Theorem Proving

- High abstraction and expressive notation
- Powerful logic and reasoning, e.g., induction
- Can exploit hierarchy and regularity, puts user in control
- Can be customized with tactics (programs that build larger proofs steps from basic ones)
- Useful for specifying and verifying parameterized (generic) datapath-dominated designs
- Unrestricted applications (at least theoretically)

Conclusions

Limitations of Theorem Proving:

- Interactive (under user guidance): use many lemmas, large numbers of commands
- Large human investment to prove small theorems
- Usable only by experts: difficult to prove large / hard theorems
- Requires deep understanding of the both the design and HOL (while-box verification)
- must develop proficiency in proving by working on simple but similar problems.
- Automated for narrow classes of designs