Formal Methods & Verification
Midterm Exam

Instructor: Farn Wang
Class hours: 9:10-12:00 Wednesday Course Nr. 921 U7600
Room: BL 103 Fall 2009

Student name: Student ID:

1. We have an 8×8 chess board and 8 chess queens. The 8-queens puzzle is to place the 8 chess queens on the chess board with the following restrictions.
   1a. Each chess queen is in one of the 8×8 positions.
   1b. No two queens are in the same row.
   1c. No two queens are in the same column.

   Suppose we have the following propositions for i∈[1,8] and j∈[1,8].

   q(i,j): true if and only if a queen is in row i and column j.

   Please use the above propositions to define the solutions of the 8-queens puzzle. (10pts/10)

(1): $\bigwedge_{i\in[1,8]}\left[\bigvee_{j\in[1,8]}\left[q(i,j)\wedge\left[\bigwedge_{k\in[1,8],k\neq j}[\neg q(i,k)]\right]\right]\right]$

(2): $\bigwedge_{j\in[1,8]}\left[\bigvee_{i\in[1,8]}\left[q(i,j)\wedge\left[\bigwedge_{k\in[1,8],k\neq i}[\neg q(k,j)]\right]\right]\right]$

Solutions: all board configurations that satisfy $(1)\wedge(2)$.

[Note that your answer should include the restriction “1a. Each chess queen is in one of the 8x8 positions”.]

2. We have the following formula:

$((p \to (q \lor r)) \land \neg q) \to (p \to r)$

Please construct a truth table to show the formula is a tautology. (5pts/15)

p	q	r	$q \lor r$	$p \to (q \lor r)$	$\neg q$	$(p \to (q \lor r)) \land \neg q$	$p \to r$	$[(p \to (q \lor r)) \land \neg q] \to (p \to r)$
F	F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T	T
F	T	T	T	T	F	F	T	T
T	F	F	F	F	T	F	F	T
T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F	T
T	T	T	T	T	F	F	T	T

So the formula is a tautology.

3. Please use Natural Deduction to show that the formula in question 2 is a tautology. (5pts/20)

1	$(p \to (q \lor r)) \land \neg q$	*assumption*
2	p	*assumption*
3	$p \to (q \lor r)$	$\land e1$
4	$q \lor r$	$\to e2,3$
5	$\neg q$	$\land e1$
6	r	$\lor e4,5$
7	$p \to r$	$\to i2,6$
8	$[(p \to (q \lor r)) \land \neg q] \to (p \to r)$	$\to i1,7$

So the formula is a tautology.

4. Please use the tableau method to prove that the formula in question 2 is a tautology. (10pts/30)

$(p \to (q \vee r)) \wedge \neg q$

$\neg(p \to r)$

$p \to (q \vee r)$

$\neg q$

p

$\neg r$

$\swarrow \qquad \searrow$

$\neg p \qquad\qquad q \vee r$

$\times \qquad\quad \swarrow \qquad \searrow$

$q \qquad\qquad r$

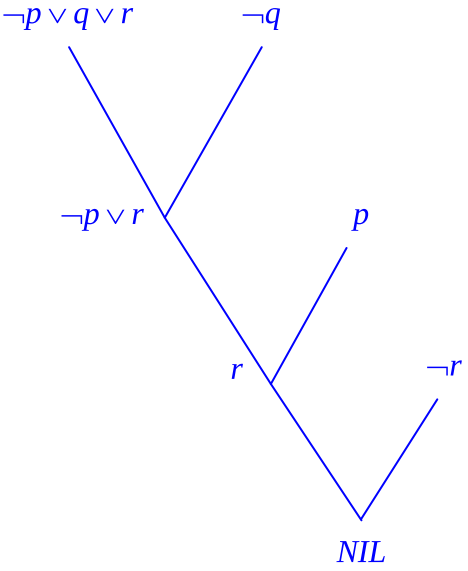
$\times \qquad\qquad \times$

Since $\neg\left[\left[(p \to (q \vee r)) \wedge \neg q\right] \to (p \to r)\right]$ is not satisfiable, the original formula is a tautology.

5. Please construct a resolution tree to show that the formula in question 2 is a tautology. (10pts/40)

$$\neg\left[\left[\left(p \to (q \vee r)\right) \wedge \neg q\right] \to (p \to r)\right]$$
$$\equiv \left[\left(p \to (q \vee r)\right) \wedge \neg q\right] \wedge \neg(p \to r)$$
$$\equiv \left[\left(\neg p \vee (q \vee r)\right) \wedge \neg q\right] \wedge (p \wedge \neg r)$$
$$\equiv (\neg p \vee q \vee r) \wedge \neg q \wedge p \wedge \neg r$$

$4\ clauses: \neg p \vee q \vee r,\ \neg q,\ p,\ \neg r$



Since $\neg\left[\left[\left(p \to (q \vee r)\right) \wedge \neg q\right] \to (p \to r)\right]$ is not satisfiable, the original formula is a tautology.

6. Please use the DPLL algorithm to show that the formula in question 2 is a tautology. (10pts/50)

$$\neg\left[\left[\left(p\to(q\vee r)\right)\wedge\neg q\right]\to(p\to r)\right]$$
$$\equiv\left[\left(p\to(q\vee r)\right)\wedge\neg q\right]\wedge\neg(p\to r)$$
$$\equiv\left[\left(\neg p\vee(q\vee r)\right)\wedge\neg q\right]\wedge(p\wedge\neg r)$$
$$\equiv(\neg p\vee q\vee r)\wedge\neg q\wedge p\wedge\neg r$$

$4\ \textit{clauses}:\ \{\neg p\vee q\vee r\},\ \{\neg q\},\ \{p\},\ \{\neg r\}$

$\underline{\textit{Unit propagation}}\ \textit{with}\ p=\textit{true}, q=\textit{false}, r=\textit{false}:$

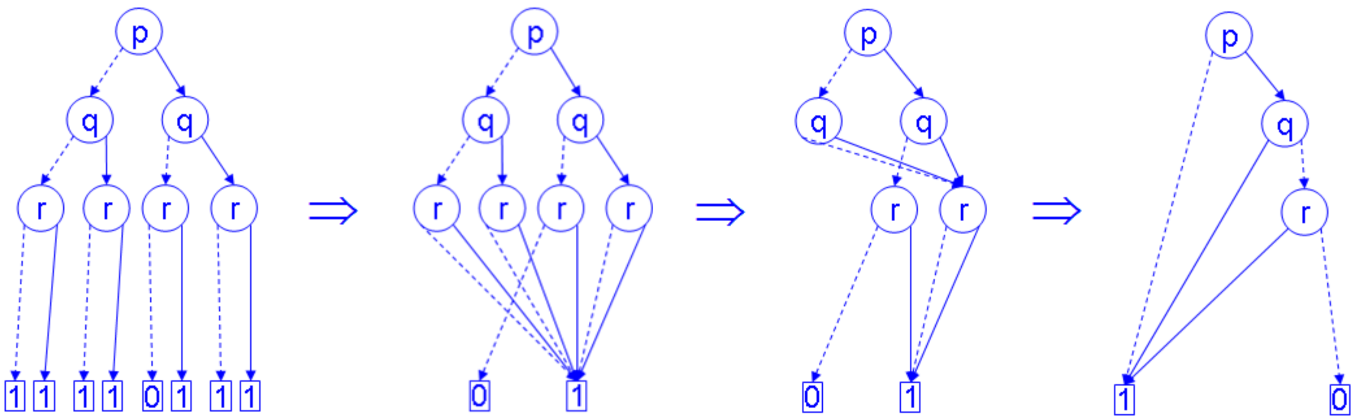
$\Rightarrow\{\ \},\ \{\textit{true}\},\ \{\textit{true}\},\ \{\textit{true}\}$

$\Rightarrow\{\ \}$

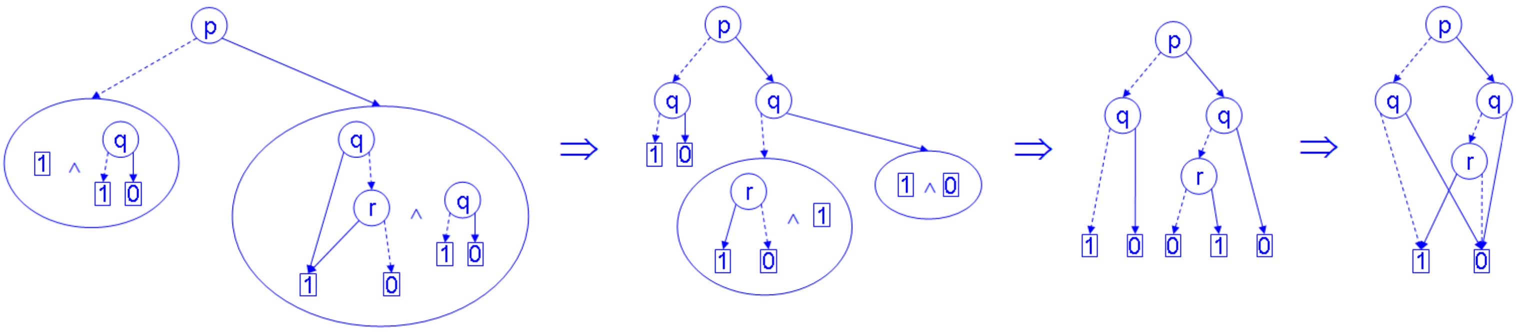
Since $\neg\left[\left[\left(p\to(q\vee r)\right)\wedge\neg q\right]\to(p\to r)\right]$ is not satisfiable, the original formula is a tautology.

7. Assume the variable ordering in BDDs is p→q→r.

7a. Please construct the BDD of p→(q∨r). (10pts/60)

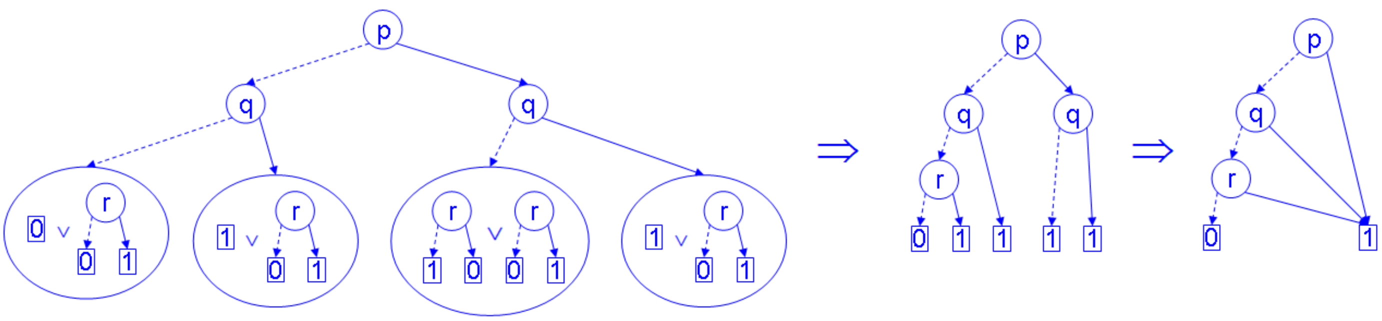


7b. Please construct the BDD of (p→(q∨r)) ∧ ¬q. (10pts/70)



7c. Please construct the BDD of ((p→(q∨r)) ∧ ¬q) → r. (10pts/80)

$$\left[\left(p\to(q\vee r)\right)\wedge\neg q\right]\to r\equiv\neg\left[\left(p\to(q\vee r)\right)\wedge\neg q\right]\vee r$$



8. We have the following grammar for tree growth.

T ::= S F | S S T T

S ::= K L L K

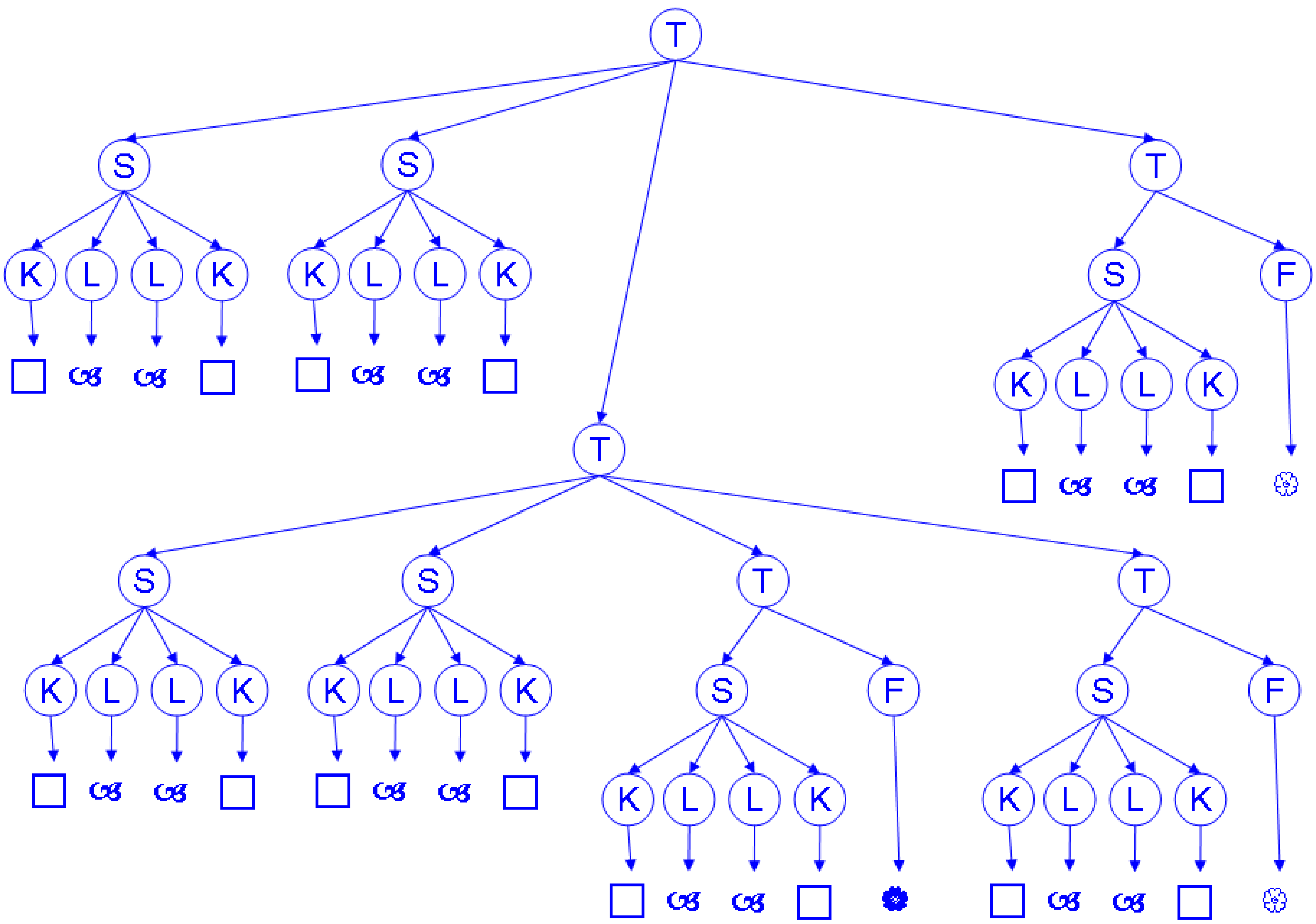
F ::= ❀ | ✿

K ::= □

L ::= ☙

Please draw the derivation tree of the following sentence. (10pts/90)

□☙☙□□☙☙□□☙☙□□☙☙□□☙☙□✿□☙☙□❀□☙☙□❀



9. Please draw a state-transition graph for the following vending machine M that accepts nickels, dimes, and quarters. M accepts changes until 25 cents have been put in. It gives changes back for any amount greater than 25 cents. Then the customer can push buttons to receive a cola or a chocolate bar. If the machine detects a fake coin, it also returns the coin immediately and tells the customer that the coin is a fake.

   Please use *!e* for an output event of type e and *?e* for an input event of type e. (10pts/100)