Control System: Homework 03 for Units 33, 34, 35, 36: Dynamic Response

Assigned: March 27, 2020
Due: April 10, 2020 (noon)

1. (Time domain specification)

3.27 For the unity feedback system shown in Fig. 3.55, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using Matlab.

![Diagram showing unity feedback system](image)

Figure 3.55  Unity feedback system for Problem 3.27

Solution:

\[
\frac{Y(s)}{R(s)} = \frac{K}{s + a} \cdot \frac{10}{s + 50} = \frac{10K}{(s + a)(s + 50) + 10K} \\
= \frac{10K}{s^2 + (50 + a)s + 10K} \approx \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}
\]

given \( R(s) = \frac{1}{s} \), \( M_p \leq 30\% \), \( t_s \leq 0.2 \) sec,

\[
M_p = e^{-\delta\sqrt{1-s^2}} \Rightarrow \delta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.35
\]

\[
e^{-\delta\omega_n} = 0.02 \text{ for 2\% settling time.}
\]

\[
t_s \leq 3.94/\delta\omega_n = 0.2 \Rightarrow \omega_n = 55.85
\]

\[
2\delta\omega_n = 50 + a \Rightarrow a = -10.9
\]

\[
\omega_n^2 = 10K \Rightarrow K = 311.92
\]
2. (Effect of zeros and additional poles)

3.42 Sketch the step response of a system with the transfer function

\[ G(s) = \frac{s/2 + 1}{(s/40 + 1)(s/2)^2 + s/4 + 1}. \]

Justify your answer on the basis of the locations of the poles and zeros. (Do not find the inverse Laplace transform.) Then compare your answer with the step response computed using Matlab.

Solution:

From the given system, we notice that the real pole is a factor of away from the pair of complex poles. Hence the response of the system is dominated by the complex pair of poles.

i.e. \( G(e) \approx \frac{s/5 + 1}{(s/2)^2 + (s/2) + 1} \)

\[ 1 + GH(s) = 0 \Rightarrow 1 + \frac{s/5 + 1}{(s/2)^2 + (s/2) + 1}.1 = 0 \]

\[ \Rightarrow s^2/4 + s/2 + 1 + s/5 + 1 = 0 \]

\[ 5s^2 + 10s + 20 + 4s + 20 = 0 \]

\[ \Rightarrow 5s^2 + 14s + 40 = 0 \]

\[ s^2 + 2.88 + 8 = 0 \]

\[ \approx s^2 + 20\delta om + om^2 = 0 \]

\[ 2\delta om = 2.8 \& om^2 = 8 \]

\[ \Rightarrow \delta = 0.5 \]

At \( \delta = 0.5 \& \omega_n = 2.82 \), we have

\[ MP = e^{\frac{s}{\sqrt{1-\delta^2}}} = 0.163 = 16.3\% \quad \text{i.e., an overshoot of 16%}. \]
3. (Effect of zeros and additional poles)

3.47 Consider the two nonminimum-phase systems.

\[ G_1(s) = -\frac{2(s - 1)}{(s + 1)(s + 2)}, \quad (3.98) \]
\[ G_2(s) = \frac{3(s - 1)(s - 2)}{(s + 1)(s + 2)(s + 3)}. \quad (3.99) \]

(a) Sketch the unit-step responses for \( G_1(s) \) and \( G_2(s) \), paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, \( m \) zeros and \( n \) poles, where \( m < n \)). Let \( y(t) \) denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the “wrong” direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an odd number of real RHP zeros.

Solution:

(a) For \( G_1(s) \):

\[ Y_1(s) = \frac{1}{s} G_1(s) = \frac{-2(s - 1)}{s(s + 1)(s + 2)}, \]
\[ H(s) = k \prod_{i} (s - z_i) \prod_{i} (s - p_i), \]
\[ R_{p_i} = \lim_{s \to p_i} [(s - p_i)H(s)] = \lim_{s \to p_i} k \frac{\prod_{i} (s - z_i) \prod_{i \neq j} (s - p_i)}{\prod_{i \neq i} (s - p_i)} = k \frac{\prod_{i} (p_i - z_j)}{\prod_{i \neq i} (p_i - p_i)}. \]

Each factor \((p_i - z_j)\) or \((p_i - p_i)\) can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to \( p_i \) and coming from \( z_j \) or \( p_i \) respectively.

The method for calculating the residue at a pole \( p_i \) is:

1. Draw vectors from the rest of the poles and from all the zeros to the pole \( p_i \).

2. Measure magnitude and phase of these vectors.

3. The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.
In our problem:

\[
Y_1(s) = \frac{-2(s - 1)}{s(s + 1)(s + 2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s + 1)} + \frac{R_{-2}}{(s + 2)} = \frac{1}{s} - \frac{4}{s + 1} + \frac{3}{s + 2},
\]

\[
y_1(t) = 1 - 4e^{-t} + 3e^{-2t}.
\]

Problem 3.47: Step response for a non-minimum phase system with one real RHP zero.

For \( G_2(s) \):

\[
Y_2(s) = \frac{3(s - 1)(s - 2)}{s(s + 1)(s + 2)(s + 3)} = \frac{1}{s} + \frac{-9}{(s + 1)} + \frac{18}{(s + 2)} + \frac{-10}{(s + 3)},
\]

\[
y_2(t) = 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}.
\]
Problem 3.47: Step response of a non-minimum phase system with two real zeros in the RHP.

(b) The first system presents an “undershoot”. The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In \( y_1(t) \): dominant at \( t = 0 \) the term \(-4e^{-t}\)

In \( y_2(t) \): dominant at \( t = 0 \) the term \( 18e^{-2t} \)

(c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain \( G(0) = 1 \) Since the system is stable, \( y(\infty) = G(0) = 1 \), and it is reasonable to assume \( y(\infty) \neq 0 \). Let us denote the pole-zero excess as \( r = n - m \). Then, \( y(t) \) and its \( r - 1 \) derivatives are zero at \( t = 0 \), and \( y''(0) \) is the first non-zero derivative. The system has an undershoot
if \( y'(^0)y(\infty) < 0 \). The transfer function may be re-written as

\[
G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{\alpha_j}{z_i})}{\prod_{i=1}^{n+p}(1 - \frac{\alpha_i}{p_i})}
\]

The numerator terms can be classified into three types of terms:

1. The first group of terms are of the form \((1 - \alpha_i s)\) with \(\alpha_i > 0\).
2. The second group of terms are of the form \((1 + \alpha_i s)\) with \(\alpha_i > 0\).
3. Finally, the third group of terms are of the form \((1 + \beta_i s + \alpha_i s^2)\) with \(\alpha_i > 0\), and \(\beta_i\) could be negative.

However, \(\beta_i^2 < 4\alpha_i\), so that the corresponding zeros are complex.

All the denominator terms are of the form (2), (3), above. Since,

\[
y'(^0) = \lim_{s \to \infty} s^r G(s)
\]

it is seen that the sign of \(y'(^0)\) is determined entirely by the number of terms of group 3 above. In particular, if the number is odd, then \(y'(^0)\) is negative and if it is even, then \(y'(^0)\) is positive. Since \(y(\infty) = G(0) = 1\), then we have the desired result.

References
4. (Stability)

3.52 A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the amplitude of the time response to double (see Fig. 3.65), given some nonzero initial condition.

(a) For a first-order system, show that the time to double is

\[ \tau_2 = \frac{\ln 2}{p} \]

where \( p \) is the pole location in the RHP.

(b) For a second-order system (with two complex poles in the RHP), show that

\[ \tau_2 = \frac{\ln 2}{-\zeta \omega_n} \]

![Figure 3.65 Time to double](image)

Solution:

(a) First-order system, \( H(s) \) could be:

\[ H(s) = \frac{k}{(s - p)} \]

\[ h(t) = \mathcal{L}^{-1} [H(s)] = ke^{pt}, \]

\[ h(\tau_0) = ke^{p\tau_0}, \]

\[ h(\tau_0 + \tau_2) = 2h(\tau_0) = ke^{p(\tau_0 + \tau_2)}, \]

\[ \implies 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2}, \]

\[ \implies \tau_2 = \frac{\ln 2}{p}. \]
\[
|f_0| = -y_0 \frac{e^{\omega_n |\zeta|t}}{\sqrt{1 - |\zeta|^2}},
\]
\[
|f_0| = -y_0 \frac{e^{\omega_n |\zeta|\tau_0}}{\sqrt{1 - |\zeta|^2}},
\]
\[
|f_0 + f_2| = -y_0 \frac{e^{\omega_n |\zeta|(\tau_0 + \tau_2)}}{\sqrt{1 - |\zeta|^2}} = 2|f_0|
\]

\[
\Rightarrow e^{\omega_n |\zeta|\tau_2} = 2
\]
\[
\Rightarrow \tau_2 = \frac{\ln 2}{\omega_n |\zeta|} = \frac{\ln 2}{-\omega_n \zeta} \quad (\zeta \leq 0)
\]

Note: This problem shows that \( \sigma = \omega_n |\zeta| \) (the real part of the poles) is inversely proportional to the time to double.
The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

(b) Second-order system:

\[
y(t) = y_0 \frac{e^{\omega_n |\zeta|t}}{\sqrt{1 - |\zeta|^2}} \sin \left(\omega_n \sqrt{1 - |\zeta|^2}t + \cos^{-1} \zeta \right),
\]

where

\[
\cos^{-1} \zeta = \cos^{-1} |\zeta| + \pi
\]

\[
\Rightarrow y(t) = y_0 \frac{e^{\omega_n |\zeta|t}}{\sqrt{1 - |\zeta|^2}} (-1) \sin \left(\omega_n \sqrt{1 - |\zeta|^2}t + \cos^{-1} |\zeta| \right)
\]

Note: Instead of working with a negative \( \zeta \), everything is changed to \( |\zeta| \).
5. (Stability)

3.56 The transfer function of a typical tape-drive system is given by

\[ KG(s) = \frac{K(s + 4)}{s(s + 0.5)(s + 1)(s^2 + 0.4s + 4)} \]

where time is measured in milliseconds. Using Routh’s stability criterion, determine the range of \( K \) for which this system is stable when the characteristic equation is \( 1 + KG(s) = 0 \).

Solution:

(a) System Characteristic equation: \( 1 + GH(s) = 0 \)

\[
1 + \frac{k(s + 4)}{s(s + 1)(s + 2)} = 0 \Rightarrow s^3 + 3s^2 + (2 + k)s + 4k
\]

<table>
<thead>
<tr>
<th>( s^3 )</th>
<th>( 1 )</th>
<th>( 2 + k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>( 3 )</td>
<td>( 4k )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( \frac{6 - k}{3} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( 4k )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

For stability

\[ 4k > 0 \quad \& \quad \frac{6 - k}{3} > 0 \]

\[ \Rightarrow k > 0 \quad \Rightarrow k < 6 \]

\[ \Rightarrow 0 < k < 6 \]

(b) For system to oscillate, there must be boles on \( jw \) exis.

Then \( A(s)3s^2 + 4k = 0 \Rightarrow 38^2 + 4(s) = 0 \Rightarrow s = \pm 2.5 \)

Frequency of oscillation = \( W_n \)

At \( k = 6 \). System oscillates

\[ \Rightarrow \text{Then } w_n = 2.5 \text{ rad/ sec}. \]