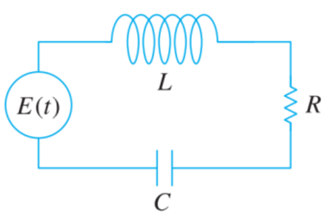
Fall 2019

微分方程
Differential Equations

Unit 01.3
DEs as Mathematical Models



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

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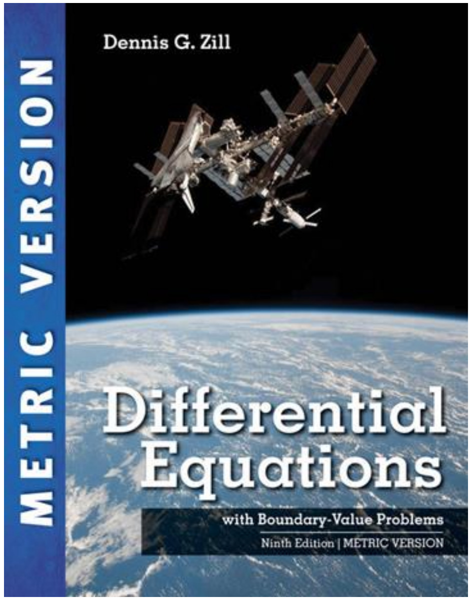
NTU-EE

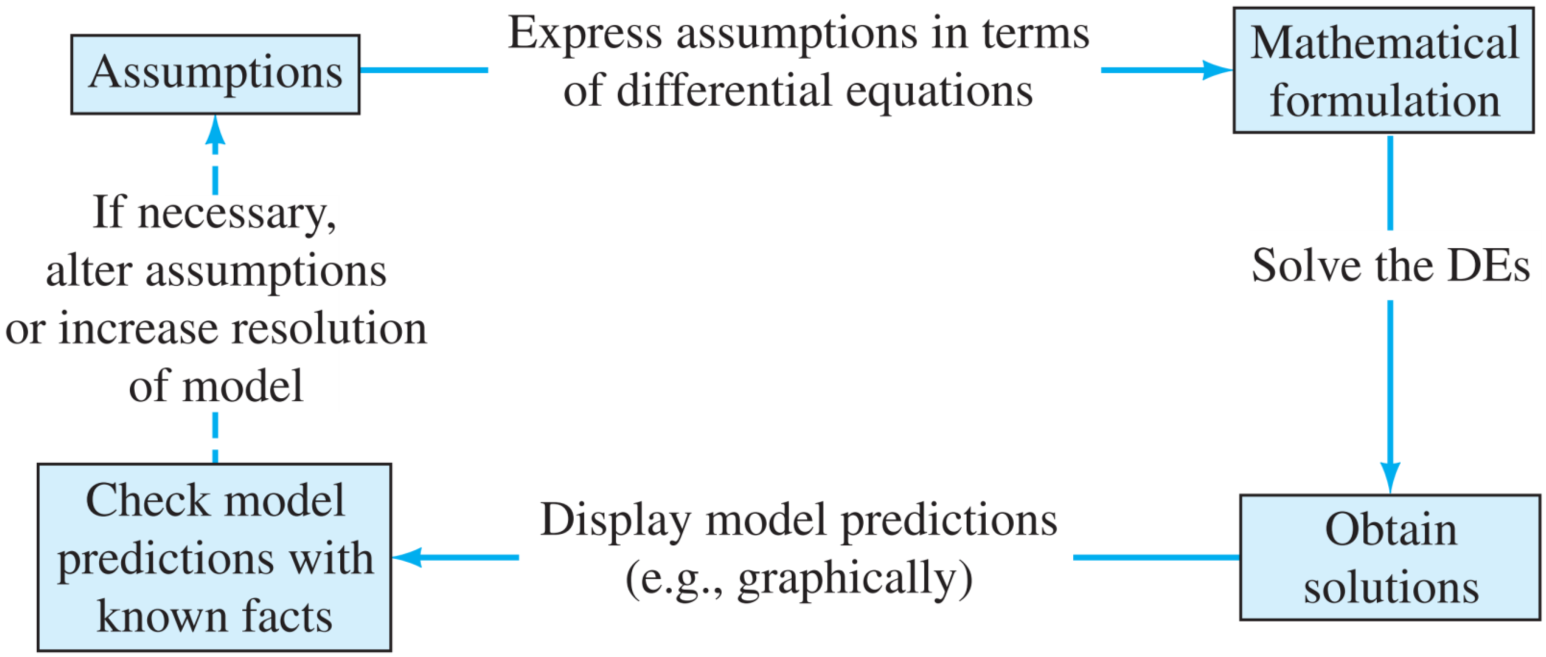
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

- The mathematical description of a system or a phenomenon is called a **mathematical model** and is constructed with certain goals in mind.
- Construction of a mathematical model of a system starts with:
  - Identify the **variables** for changing the system and specify the **level of resolution** of the model
  - Reasonable assumptions, or hypotheses, and theorems or empirical laws

- Modeling process



- Variables and Constants:
  - $P(t)$: the total population at time t
  - k: a constant of proportionality

- Model:

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = k\,P \qquad \text{--- (1)}$$

- Variables and Constants:
  - $A(t)$: the amount of the substance remaining at time t
  - k: a constant of proportionality

- Model:

$$\frac{dA}{dt} \propto A \quad \textbf{or} \quad \frac{dA}{dt} = kA \qquad \text{--- (2)}$$

- Note:

A single DE can serve as a mathematical model for many different phenomena.

- Variables and Constants:
  - $T(t)$: the temperature of a body at time t
  - T_m: the temperature of the surrounding medium
  - k: a constant of proportionality

- Model:

$$\frac{dT}{dt} \propto T - T_m \quad \text{or} \quad \frac{dT}{dt} = k\,(T - T_m) \qquad \text{--- (3)}$$

- Variables and Constants:
  - $x(t)$: # of people who have contracted the disease
  - $y(t)$: # of people who have not yet been exposed
  - k: a constant of proportionality

- Model:

$$\frac{dx}{dt} = k\,x\,y \qquad \text{--- (4)}$$

$$\frac{dx}{dt} = k\,x\,(n + 1 - x) \qquad \text{--- (5)}$$

$$IF:\ x + y = n + 1$$

- Variables and Constants:
  - X: the amount of chemical C formed at time t
  - α, β: the amounts of chemicals A and B at $t = 0$ (the initial amounts)
  - k: a constant of proportionality
- Model:

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X) \qquad \text{--- (6)}$$

$$CH_3Cl + NaOH \longrightarrow CH_3OH + NaCl$$

- Variables and Constants:
  - $A(t)$: the salt in kilograms in the tank at time t
  - R_{in}: input rate of salt in kilograms per minute
  - R_{out}: output rate of salt
  - Concentration of salt inflow/outflow
  - Input/output rate of brine

- Model:

$$\frac{dA}{dt} = R_{in} - R_{out} \quad \text{--- (7)}$$

$$\frac{dA}{dt} = 2.5 - \frac{A}{100} \quad \text{or} \quad \frac{dA}{dt} + \frac{1}{100} A = 2.5 \quad \text{--- (8)}$$

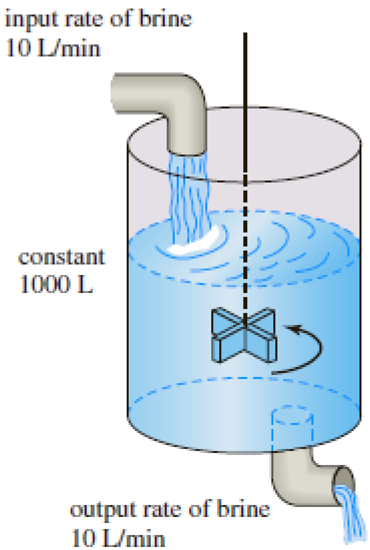
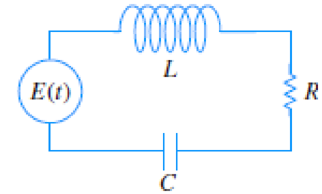


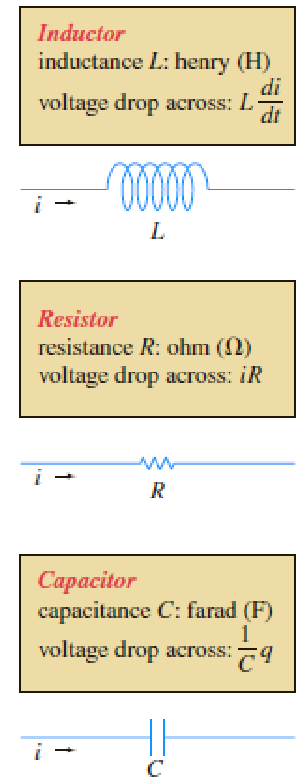
Figure 1.3.2: Mixing tank

Series Circuits

- **Variables and Constants:**
  - $i(t)$: The current in a circuit at time t
  - $q(t)$: The charge on a capacitor at time t
  - L, R, C: Inductance, resistance, and capacitance



(a) *LRC*-series circuit



(b)

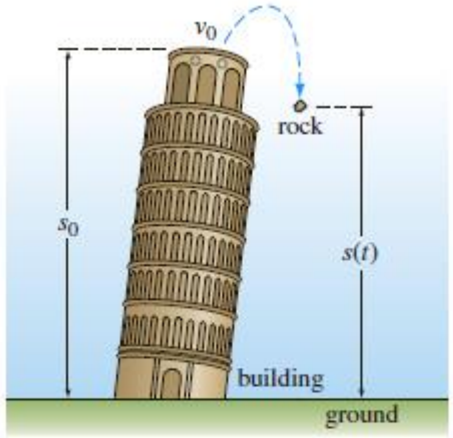
- **Model:**

$$L\frac{di}{dt} = L\frac{d^2q}{dt^2} \qquad iR = R\frac{dq}{dt} \qquad \frac{1}{C}q$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

--- (11)

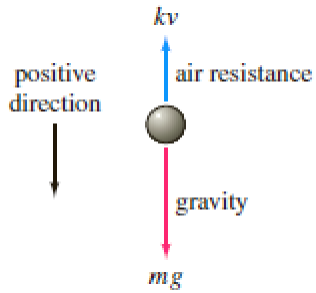
- Variables and Constants:
  - $s(t)$: Position of rock relative to the ground at time t



- Model:

$$m\frac{d^2s}{dt^2} = -mg \quad \text{or} \quad \frac{d^2s}{dt^2} = -g \quad (12)$$

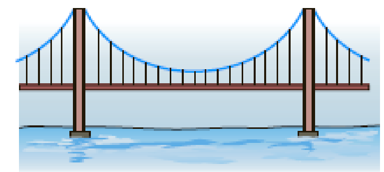
$$\frac{d^2s}{dt^2} = -g, \qquad s(0) = s_0, \qquad s'(0) = v_0 \quad (13)$$



$$m\frac{dv}{dt} = mg - kv \quad (14)$$

$$m\frac{d^2s}{dt^2} = mg - k\frac{ds}{dt} \quad \text{or} \quad m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg \quad (15)$$

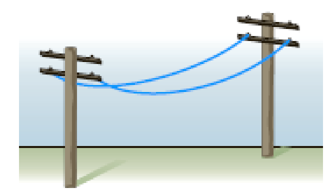
- Variables and Constants:
    - $s(t)$: Position of rock relative to the ground at time t



(a) suspension bridge cable

- Model:

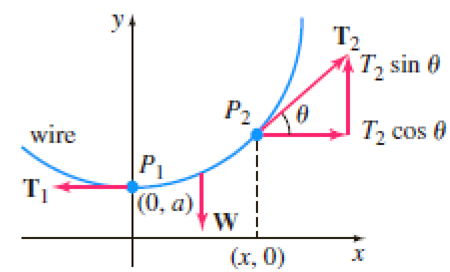
$$T_1 = T_2 \cos\theta \quad \text{and} \quad W = T_2 \sin\theta$$



(b) telephone wires

$$\tan\theta = \frac{W}{T_1} \quad \text{and} \quad dy/dx = \tan\theta$$

$$\frac{dy}{dx} = \frac{W}{T_1} \qquad (16)$$



- $\dfrac{dy}{dx} = f(\, x,\, y\,)$

$$\Rightarrow y = \Phi(x) \qquad\qquad y(x_0) = c_0$$

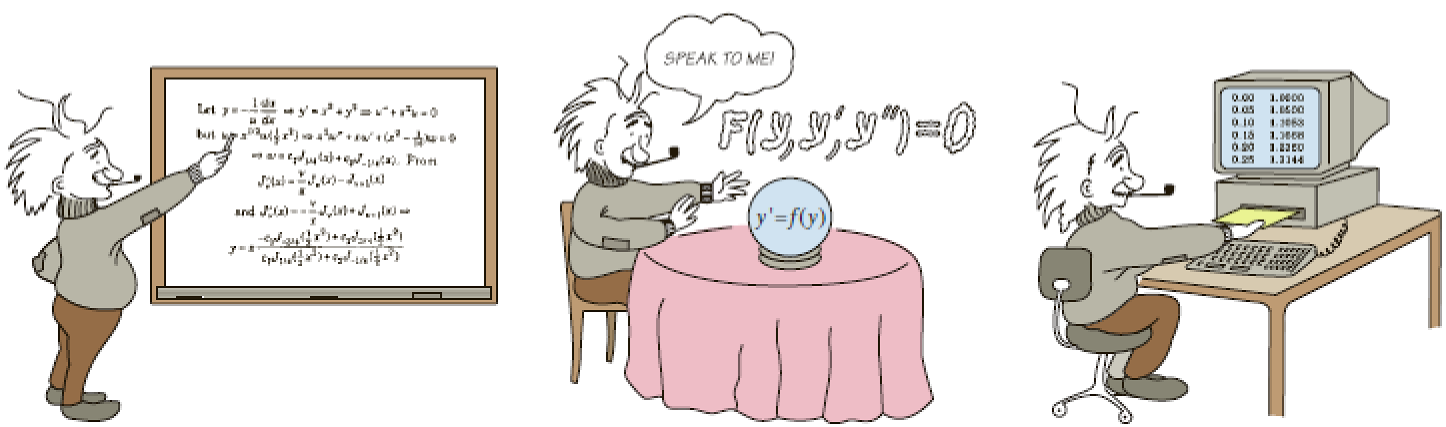
- $a_2(x)\,\dfrac{d^2 y(x)}{dx^2} + a_1(x)\,\dfrac{dy(x)}{dx} + a_0(x)\, y(x) = g(x)$

$$\Rightarrow y = \Phi(x) \qquad \begin{cases} y\,(x_0) & = & p_0 \\ y'\,(x_0) & = & v_0 \end{cases}$$

$$\begin{cases} y\,(x_1) & = & c_1 \\ y\,(x_2) & = & c_2 \end{cases}$$

Qualitative Analysis vs Numerical Analysis

- Does the DE actually have solutions?
- If a solution of the DE exists and satisfies an initial condition, is it the only such solution?
- What are some of the properties of the unknown solutions?
- What can we say about the geometry of the solution curves?
- Can we somehow approximate the values of an unknown solution?



(a) analytical (b) qualitative (c) numerical

Lectures and Exams

- Midterm Exam
  - 1-1, 1-2, 1-3 (Introduction)
  - 2-1, 2-2, 2-3, 2-4, 2-5, 2-6 (First Order DE - Math)
  - 3-1, 3-2, 3-3 (First Order DE - Modeling)
  - 4-1, 4-2, 4-3, 4-4, 4-5 (Higher Order DE - Math)
- Final Exam
  - 4-6, 4-7 (Higher Order DE - Math)
  - 5-1 (Higher Order DE - Modeling)
  - 7-1, 7-2, 7-3, 7-4, 7-5, 7-6 (Transform – Laplace)
  - 11-1, 11-2, 11-3 (Transform – Fourier)
  - 12-1 (Partial DE)