Fall 2019

微分方程
Differential Equations

Unit 02.3
Linear Equations

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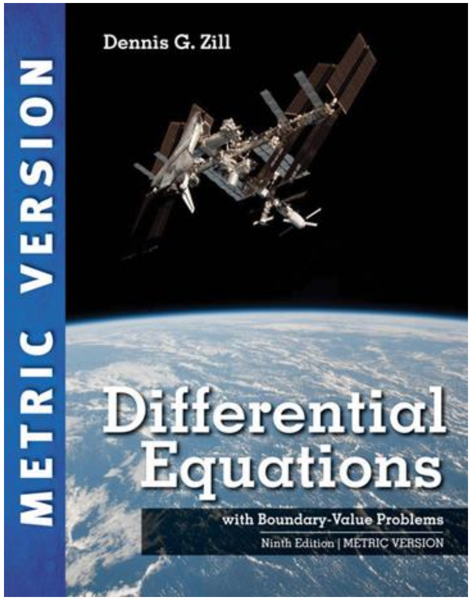
$$y(x) = y_c(x) + y_p(x)$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- 2.1: Solution Curves without a Solution
  - 2.1.1: Direction Fields
  - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- **2.3: Linear Equations**
- 2.4: Exact Equations
- 2.5: Solutions by Substitutions
- 2.6: A Numerical Method

Definition 2.3.1: Linear Equation

- A first-order DE of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(\,x\,)$$

- is said to be a linear equation in the variable y.

$$g(\,x\,) = 0$$
- homogeneous

$$g(\,x\,) \neq 0$$
- nonhomogeneous

- Standard form:

$$\frac{dy}{dx} + P(x)\,y = f(\,x\,)$$

- **Standard form:**

$$\frac{dy}{dx} + P(x)\, y = f(\, x\,)$$

$$(a) \quad \frac{dy_c}{dx} + P(x)\, y_c = 0$$

$y_c(x)$:

- Complementary function
  homogeneous solution

$$(b) \quad \frac{dy_p}{dx} + P(x)\, y_p = f(\, x\,)$$

$y_p(x)$:

- Particular solution of
  nonhomogeneous equation

$$\Rightarrow \quad y(x) = y_c(x) + y_p(x)$$

- Standard form:

$$\frac{dy}{dx} + P(x)\,y = f(\,x\,)$$

$$\frac{dy}{dx} + P(x)\,y = \frac{d(y_c + y_p)}{dx} + P(x)\,(y_c + y_p)$$

$$= \frac{d(y_c)}{dx} + P(x)\,(y_c)$$

$$+ \frac{d(y_p)}{dx} + P(x)\,(y_p)$$

$$= 0 \qquad + f(x)$$

$$= f(x)$$

Method of Solution: Homogeneous Solution

$$(a) \quad \frac{dy_c}{dx} + P(x)\, y_c = 0$$

Method of Solution: Particular Solution

$$(b) \quad \frac{dy_p}{dx} + P(x)\, y_p = f(\,x\,)$$

Example 1: Solving a Linear DE

$$\frac{dy}{dx} - 3y = 0$$

Example 2: Solving a Linear DE

$$\frac{dy}{dx} - 3y = 6$$

Example 3: General Solution

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

Singular Points

$$a_1(x) \frac{dy}{dx} + a_0(x)\, y = g(\, x\,) \qquad \frac{dy}{dx} + P(x)\, y = f(\, x\,)$$

$\Rightarrow \quad a_1(x) = 0 \qquad \rightarrow \quad x$: Singular Points

$$\Rightarrow \quad P(x) = \frac{a_0(x)}{a_1(x)}$$

$\rightarrow \quad P(x)$: discontinuous at x

Example 4: General Solution

$$(x^2 - 9)\frac{dy}{dx} + x\,y = 0$$

Example 5: An IVP

$$\frac{dy}{dx} + y = x \qquad y(0) = 4$$

Error Functions

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- Error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

- Complementary Error function

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1$$

$$\Rightarrow \begin{cases} \operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \\ \operatorname{erf}(0) = 0 \\ \operatorname{erfc}(0) = 1 \end{cases}$$

Example 7: The Error Function

$$\frac{dy}{dx} - 2xy = 2 \qquad y(0) = 1$$

Summary

$$\frac{dy}{dx} = f(\,x,\; y\,) \quad \Rightarrow \quad \frac{dy}{dx} + P(x)\,y = f(x)$$

$$\Rightarrow \quad \frac{dy}{dx} + P(x)\,y = 0$$

$$\Rightarrow \quad \frac{1}{y}\,dy = -P(x)\,dx$$

$$\Rightarrow \quad y_1(x) = e^{-\int P(x)dx}$$

$$I.F. = e^{\int P(x)dx}$$

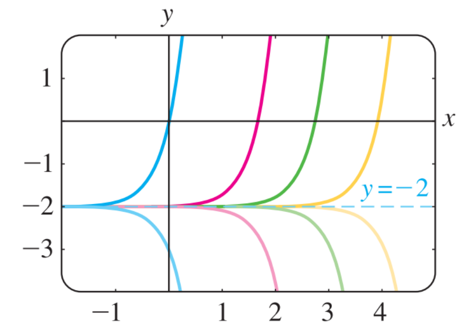
$$\Rightarrow \quad \begin{cases} y_c(x) = c\,y_1(x) \\ y_p(x) = u(x)\,y_1(x) \end{cases}$$

$$\Rightarrow \quad y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx$$

Summary

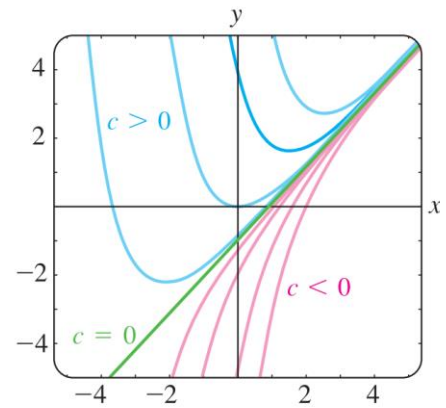
Example 2:

$$\frac{dy}{dx} - 3y = 6 \qquad y = -2 + c\,e^{3x}$$



Example 5:

$$\frac{dy}{dx} + y = x \qquad y(0) = 4 \qquad y = x - 1 + c\,e^{-x}$$



Example 7:

$$\frac{dy}{dx} - 2xy = 2 \qquad y(0) = 1 \qquad y = e^{x^2}\left[1 + \sqrt{\pi}\,\mathrm{erf}(x)\right]$$