Fall 2019

微分方程
Differential Equations

Unit 02.4
Exact Equations

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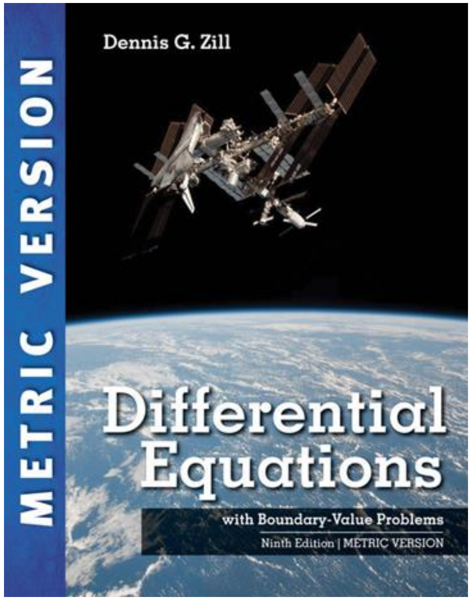
$$M(x,y)dx + N(x,y)dy = 0$$

$$\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

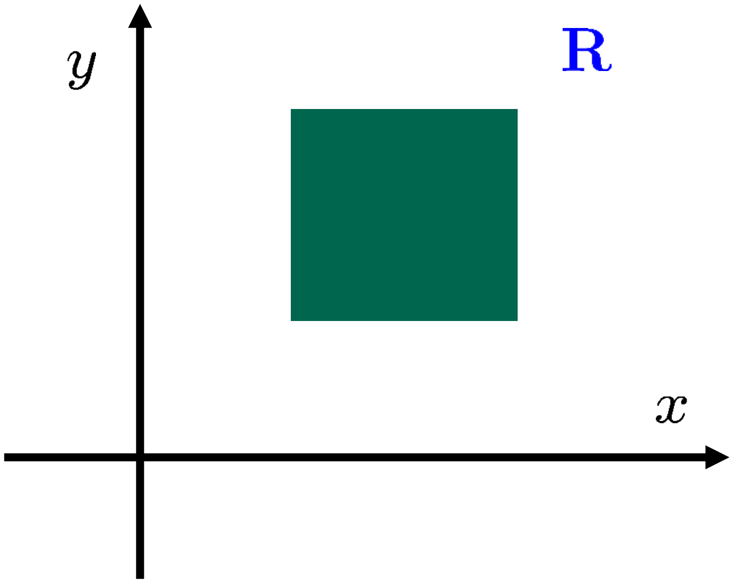


Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- 2.1: Solution Curves without a Solution
  - 2.1.1: Direction Fields
  - 2.1.2: Autonomous First-Order DEs
- 2.2: Separable Equations
- 2.3: Linear Equations
- **2.4: Exact Equations**
- 2.5: Solutions by Substitutions
- 2.6: A Numerical Method

- $z = f(x, y)$



- continuous partial derivatives in R

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

- its differential:

$$\rightarrow \quad dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- IF: $f(x, y) = c$

$$\Rightarrow \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Definition 2.4.1: Exact Equation

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- A differential expression: $M(x,y)\,dx + N(x,y)\,dy$
- is an exact differential in a region R of the $xy-$plane
- if it corresponds to the differential of some function $f(x, y)$ defined in R.

- A first-order DE of the form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

- is said to be an exact equation if the expression on the left-hand side is an exact differential.

Theorem 2.4.1: Criterion for an Exact Differential

- Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, \;\; c < y < d$.

- Then a necessary and sufficient condition that $M(x, y)\, dx \;+\; N(x, y)\, dy$ be an exact differential is:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Example 1: Solving an Exact DE

$$2xy\,dx + (x^2 - 1)\,dy = 0$$

Example 2: Solving an Exact DE

$$(e^{2y} - y\cos(xy))\,dx + (2xe^{2y} - x\cos(xy) + 2y)\,dy = 0$$

NonExact DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

$\mu(x,y)$: Integrating Factor

$$\Rightarrow \quad \mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

Exact Equation !

$$\Rightarrow \quad (\mu\, M)_y = (\mu\, N)_x$$

$$\Rightarrow \quad \mu_y\, M + \mu\, M_y = \mu_x\, N + \mu\, N_x$$

$$\Rightarrow \quad \mu_x\, N - \mu_y\, M = -\mu\, N_x + \mu\, M_y$$

$$= \mu\,(\,M_y - N_x\,)$$

NonExact DE

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- IF $\mu(x,y) = \mu(x)$ $\Rightarrow$ $\mu_x = \dfrac{d\mu}{dx}$

$$\Rightarrow \quad \frac{d\mu}{dx} N = \mu \, (\, M_y - N_x \,)$$

$$\Rightarrow \quad \frac{d\mu}{dx} = \frac{(\, M_y - N_x \,)}{N} \mu$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = \frac{(\, M_y - N_x \,)}{N} dx$$

$$\Rightarrow \quad \ln \mu = \int \frac{(\, M_y - N_x \,)}{N} dx$$

$$\Rightarrow \quad \mu(x) = e^{\int \frac{(\, M_y - N_x \,)}{N} dx}$$

NonExact DE

- IF $\mu(x,y) = \mu(y)$ $\Rightarrow$ $\mu_y = \dfrac{d\mu}{dy}$

$$\Rightarrow \quad \frac{d\mu}{dy} M = \mu \, (\, N_x - M_y \,)$$

$$\Rightarrow \quad \frac{d\mu}{dy} = \frac{(\, N_x - M_y \,)}{M} \mu$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = \frac{(\, N_x - M_y \,)}{M} dy$$

$$\Rightarrow \quad \ln \mu = \int \frac{(\, N_x - M_y \,)}{M} dy$$

$$\Rightarrow \quad \mu(y) = e^{\int \frac{(\, N_x - M_y \,)}{M} dy}$$

Example 4: A Nonexact DE Made Exact

$$x\,y\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

Summary

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \Rightarrow \quad d\Big(f(x,y)\Big) = 0$$

$$\Leftrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad f(x,y) = c$$

- Nonexact Equations: (2.4)

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\Rightarrow \mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

$$\Leftrightarrow \quad (\mu M)_y = (\mu N)_x$$

$$\Rightarrow \mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

or $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$