Fall 2019

微分方程
Differential Equations

Unit 02.5
Solutions by Substitutions

$$\frac{dy}{dx} = f(x, y)$$
$$\Rightarrow y(x) = g(x, u)$$

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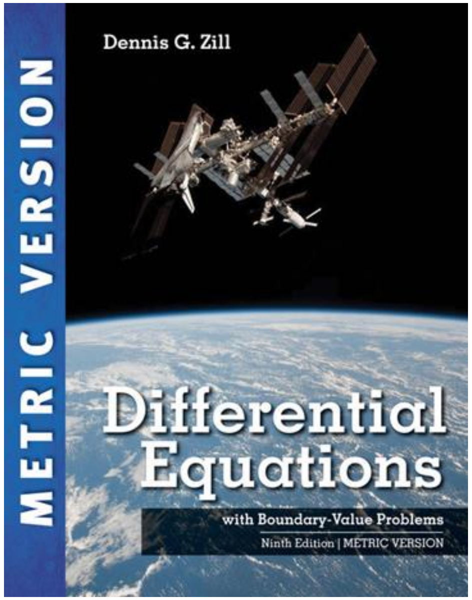

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- Transforming the DE into another DE

  by means of a substitution.

  - $\frac{dy}{dx} = f(\ x,\ y\ )$

$\Rightarrow \ y(x) = h(x)$ Assume $y(x) = g(x, u(x))$

$$\Rightarrow \frac{dy}{dx} = g_x + g_u \frac{du}{dx} = f(\ x,\ g(x,u)\ ) \Rightarrow \frac{du}{dx} = \frac{f - g_x}{g_u}$$

$$= F(x,u)$$

$$\Rightarrow u(x)$$

$$\Rightarrow y(x) = g(x,u)$$

1) Homogeneous Equations
2) Bernoulli's Equation
3) Reduction to Separation of Variables

- Homogeneous Functions of Degree ***a***

$f(x, y)$ is a homogeneous functions of degree ***a***

IF $f(tx, ty) = t^a f(x, y)$

e.g., $f(x, y) = x^3 + 3y^3$

$$f(tx, ty) = (tx)^3 + 3(ty)^3$$

$$= t^3 x^3 + 3 t^3 y^3$$

$$= t^3 (x^3 + 3 y^3)$$

$$= t^3 f(x, y)$$

Homogeneous Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M(x,y), N(x,y)$ Homogeneous Functions of Degree ***a***

$$\Rightarrow \begin{cases} M(tx,ty) = t^a M(x,y) \\ N(tx,ty) = t^a N(x,y) \end{cases}$$

$$y = u\,x \qquad u = \frac{y}{x} \qquad \Rightarrow \begin{cases} M(x,y) = x^a M(1,u) \\ N(x,y) = x^a N(1,u) \end{cases}$$

OR

$$x = v\,y \qquad v = \frac{x}{y} \qquad \Rightarrow \begin{cases} M(x,y) = y^a M(v,1) \\ N(x,y) = y^a N(v,1) \end{cases}$$

Homogeneous Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$x^a M(1,u)\,dx + x^a N(1,u)\,dy = 0$$

$$M(1,u)\,dx + N(1,u)\,dy = 0$$

$$y = u\,x \qquad dy = u\,dx + du\,x$$

$$M(1,u)\,dx + N(1,u)\,[\,u\,dx + x\,du\,] = 0$$

$$[\,M(1,u) + u\,N(1,u)\,]\,dx + x\,N(1,u)\,du = 0$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{M(1,u) + u\,N(1,u)} = 0$$

- A Separable DE

Example 1: Solving a Homogeneous DE

$$(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$$

$$\frac{dy}{dx} + P(x)\,y = f(\,x\,)\,y^n \qquad n \in \mathbf{R}$$

$\rightarrow$ $n = 0$ or 1 : linear equations

$\rightarrow$ $n \neq 0$ or 1 : $u = y^{1-n}$

$$i.e., \quad y = g(x,u) = u^{\frac{1}{1-n}}$$

$$\Rightarrow \frac{du}{dx} = F(\,x,\ u\,)$$ linear equations

Example 2: Solving a Bernoulli DE

$$x \frac{dy}{dx} + y = x^2 y^2$$

Reduction to Separation of Variables

$$\frac{dy}{dx} = f(\,A\,x + B\,y + C\,)$$

$$u = A\,x + B\,y + C, \qquad B \neq 0$$

$$\frac{du}{dx} = A + B\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right) = f(du)$$

$$\frac{du}{dx} - A = B\,f(\,du\,)$$

$$\frac{du}{dx} = A + B\,f(\,du\,)$$

$$\frac{du}{A + B\,f(\,du\,)} = dx$$

Example 3: An IVP

$$\frac{dy}{dx} = (-2x + y)^2 - 7, \qquad y(0) = 0$$

Summary

$$\frac{dy}{dx} = f(\,x,\ y\,) \quad \Rightarrow \quad y(x) = g(x, u(x))$$

$$\Rightarrow \quad \frac{du}{dx} = F(\,x,\ u\,)$$

$$\Rightarrow \quad u(x) = \cdots$$

$$\Rightarrow \quad y(x) = \cdots$$

e.g., $y = u\ x$

$$dy = du\ x + u\ dx$$

$$dy = f(\,x,\ y\,)dx \quad \Rightarrow \quad du\ x + u\ dx = f(\,x,\ u\ x\,)dx$$