Fall 2019

微分方程
Differential Equations

Unit 04.1
Preliminary Theory - Linear Equations

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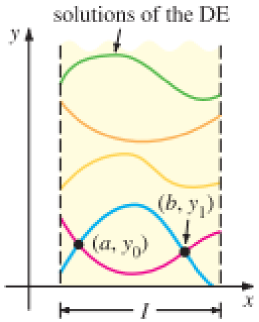
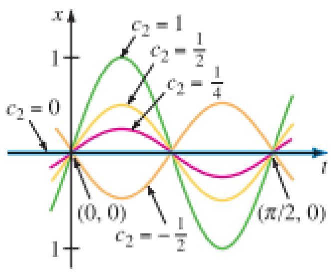
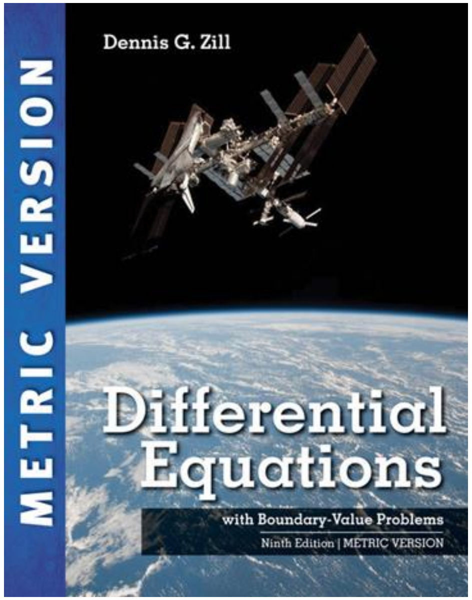
NTU-EE

Sep19 – Jan20







Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

• nth-Order Initial-Value Problems (IVP):

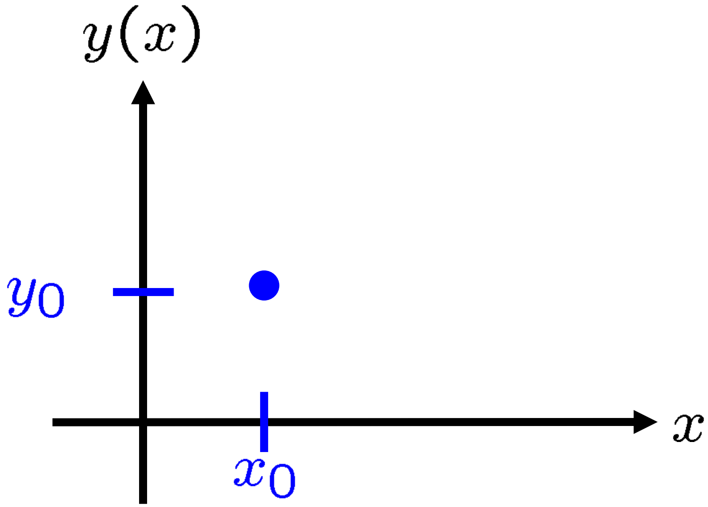
Solve

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

Subject to:

$$\begin{aligned} y(x_0) &= \\ y'(x_0) &= \\ y^{(2)}(x_0) &= \\ &\vdots \\ y^{(n-1)}(x_0) &= \end{aligned}$$

Solution:



4.1.1: Theorem 4.1.1: Existence of a Unique Solution

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ y^{(2)}(x_0) = y_2,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1}$

- Let $a_n(x),\ a_{n-1}(x),\ \cdots,\ a_1(x),\ a_0(x)$ & $g(x)$

  be

- And $a_n(x)$

- Then IVP exsits a unique solution $y(x)$ on I

4.1.1: Examples 1 & 2

$$3\, y''' + 5\, y'' - y' + 7\, y = 0 \qquad y(1) = 0,\quad y'(1) = 0,\quad y''(1) = 0$$

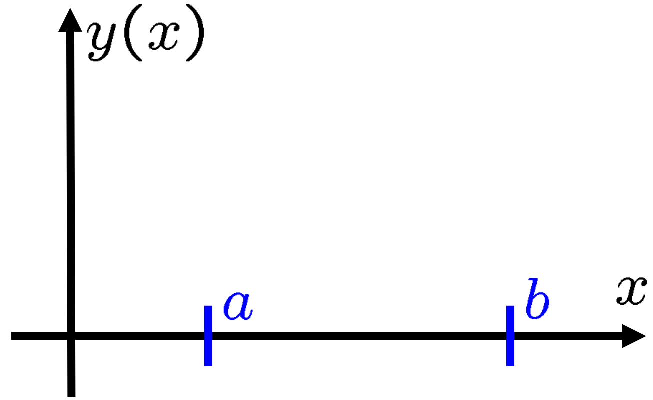
$$y'' - 4\, y = 12\, x \qquad y(0) = 4,\quad y'(0) = 1$$

$$x^2\, y'' - 2\, x\, y' + 2\, y = 6 \qquad y(0) = 3,\quad y'(0) = 1$$

- **2nd-Order Boundary-Value Problems (BVP):**

Solve
$$a_2(x)\,\frac{d^2y(x)}{dx^2} + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

Subject to:
$$\begin{cases} y(\qquad) = y_0 \\ y(\qquad) = y_1 \end{cases}$$



Other possible BCs:
$$\begin{cases} y(\qquad) = y_0 \\ y(\qquad) = y_1 \end{cases} \quad \begin{cases} y(\qquad) = y_0 \\ y(\qquad) = y_1 \end{cases} \quad \begin{cases} y(\qquad) = y_0 \\ y(\qquad) = y_1 \end{cases}$$

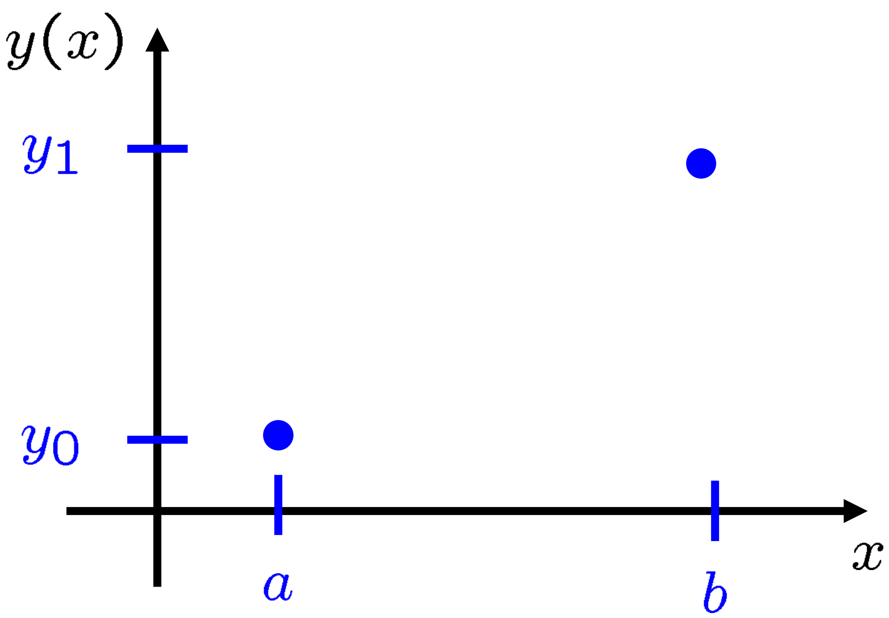
General BCs:
$$\begin{cases} \qquad y(\qquad) + \qquad y'(\qquad) = \\ \qquad y(\qquad) + \qquad y'(\qquad) = \end{cases}$$

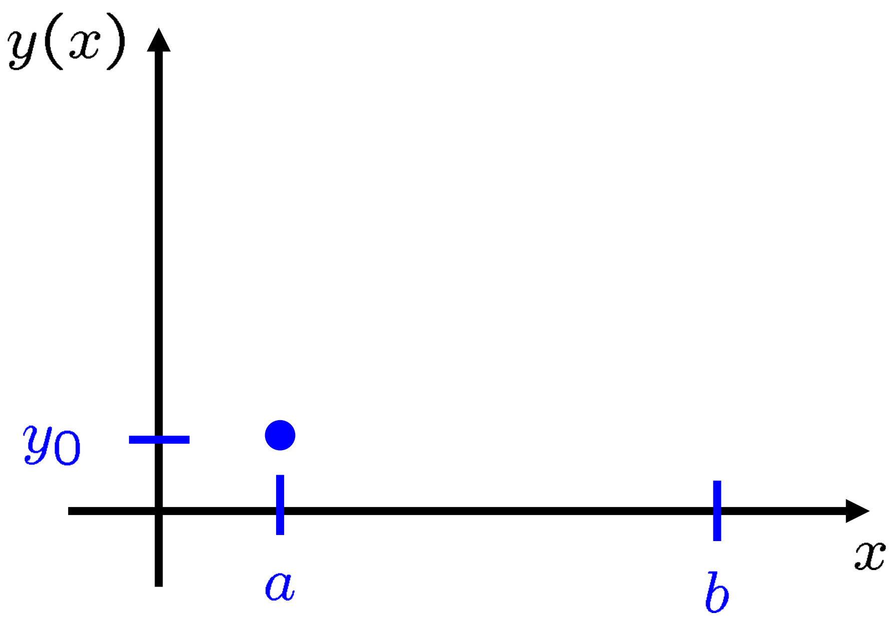
- A BVP may have:

(1) solutions

or (2) solution

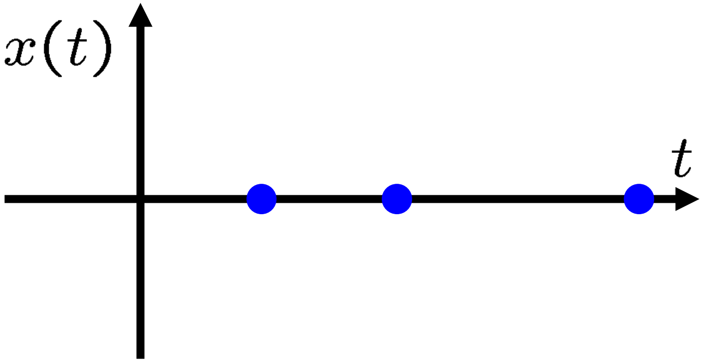
or (3) solution





4.1.1: Example 3

$$x'' + 16\,x = 0$$



$\Rightarrow$ solution:

$$(a)\ \begin{cases} x(0) = 0 \\ x(\frac{\pi}{2}) = 0 \end{cases}$$

$$(b)\ \begin{cases} x(0) = 0 \\ x(\frac{\pi}{8}) = 0 \end{cases}$$

$$(c)\ \begin{cases} x(0) = 0 \\ x(\frac{\pi}{2}) = 1 \end{cases}$$

4.1.2: Homogeneous Equations

$$a_n(x)\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + \cdots + a_1(x)\frac{dy(x)}{dx} + a_0(x)\,y(x) = 0$$

$$a_n(x)\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + \cdots + a_1(x)\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

$$g(x) \neq 0$$

- associated homogeneous equation:

$$F\left(x, y, y', \cdots, y(n)\right) = g(x)$$

$$F\left(x, y, y', \cdots, y(n)\right)$$

- In the following, we assume on some common interval I

(1) $a_i(x)$, $i = 0, 1, \cdots, n$, are

(2) $g(x)$ is

(3) $a_n(x)$

4.1.2: Differential Operators

$D\,y \triangleq$

D :

$D^2\,y \triangleq$

$D^n\,y \triangleq$

$(D + 3)\,y \triangleq$

$D+3$:

4.1.2: nth-Order Differential Operator / Polynomial Operator

$$L = \quad D^n + \quad D^{n-1} + \cdots + \quad D +$$

$$\Rightarrow L(y) =$$

for example,

$$L = (x+5)D + x$$

$$\Rightarrow L(y) =$$

- homogeneous equation: $L(y) =$
- nonhomogeneous equation: $L(y) =$

- D is a linear operator

$$\begin{cases} D \left\{ f(x) \ + \ g(x) \right\} \ = \\ \\ D \left\{ c \ f(x) \right\} \ = \end{cases}$$

or $$D \left\{ a \ f(x) \ + \ b \ g(x) \right\} \ =$$

$\Rightarrow L$ is a linear operator

$$\Rightarrow L \left\{ a \ f(x) \ + \ b \ g(x) \right\} \ =$$

- Let $y_1(x), \ y_2(x), \cdots, \ y_k(x)$ be
- Then $y(x) = \quad y_1(x) + \quad y_2(x) + \cdots + \quad y_k(x)$

  is also

  where are arbitrary constants
- Proof:

4.1.2: Example 4

$$x^3 y''' - 2 x y' + 4 y = 0$$

$$y_1(x) = x^2$$

on $(0, \infty)$

$$y_2(x) = x^2 \ln x$$

$$\Rightarrow y(x) = \quad y_1(x) + \quad y_2(x)$$

4.1.2: Def 4.1.1: Linear Dependence / Independence

- $\left\{ f_1(x),\ f_2(x), \cdots,\ f_n(x) \right\}$ is said to be

- If there exist constants

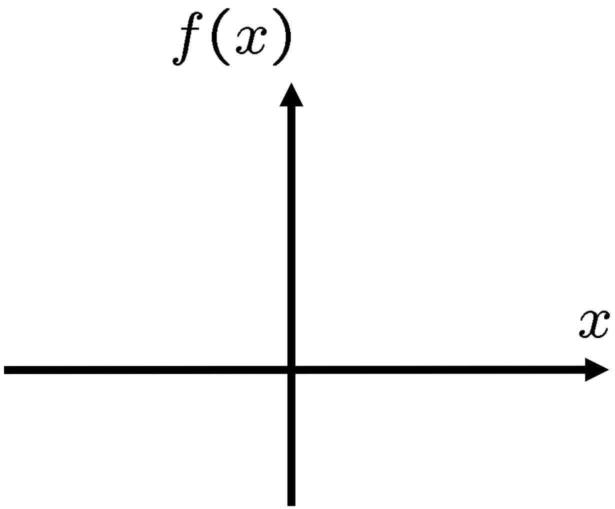
- such that $f_1(x) + \qquad f_2(x) + \cdots + \qquad f_n(x)$

- If $\left\{ f_1(x),\ f_2(x), \cdots,\ f_n(x) \right\}$ is NOT linear depedent

- Then, the set of functions is

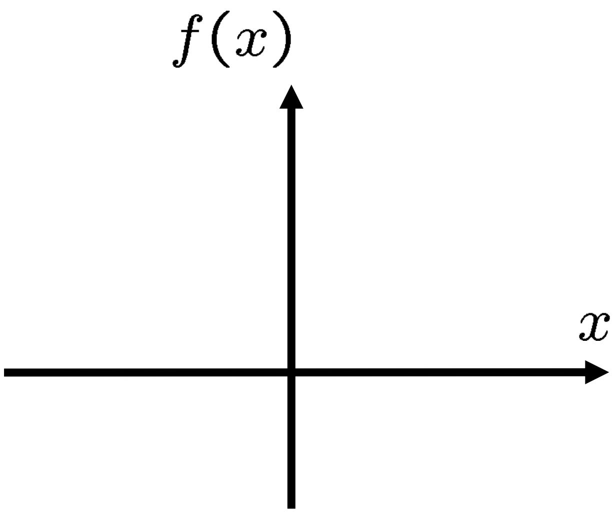
- That is,

- $\left\{ \sin(2x),\ \sin(x)\cos(x) \right\}$ $\quad I = (-\infty, \infty)$



- $\left\{ x,\ |x| \right\}$ $\quad x \in (-\infty, \infty)$

- $\left\{ x,\ |x| \right\}$ $\quad x \in [0, \infty)$



- $\left\{ x,\ |x| \right\}$ $\quad x \in (-\infty, 0]$

- Suppose each of $f_1(x),\ f_2(x), \cdots,\ f_n(x)$ has at least

- Define:

$$W(f_1, f_2, \cdots, f_n) \triangleq \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ & & \cdots & \\ \vdots & & \ddots & \\ & & \cdots & \end{bmatrix}$$

as the of $f_1(x),\ f_2(x), \cdots,\ f_n(x)$

Józef Maria Hoene-Wroński (1776 – 1853)

4.1.2: Thm 4.1.3: Criterion for Linearly Independent Solutions

- Let $y_1(x),\ y_2(x), \cdots,\ y_n(x)$ be

- Then $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is

- IFF (if and only if)

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

$$c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_n y_n'(x) = 0$$

$$\vdots$$

$$c_1 y_1''(x) + c_2 y_2''(x) + \cdots + c_n y_n''(x) = 0$$

$$c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) = 0$$

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & \cdots & y_n''(x) \\ & & \vdots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

4.1.2: Thm 4.1.3: Criterion for Linearly Independent Solutions

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0\, y = 0$$

$$a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0\, y = -\, a_n \frac{d^n y}{dx^n}$$

$$a_{n-1} \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1 \frac{dy_1}{dx} + a_0\, y_1 = -\, a_n \frac{d^n y_1}{dx^n}$$

$$a_{n-1} \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1 \frac{dy_2}{dx} + a_0\, y_2 = -\, a_n \frac{d^n y_2}{dx^n}$$

$$\vdots$$

$$a_{n-1} \frac{d^{n-1} y_n}{dx^{n-1}} + \cdots + a_1 \frac{dy_n}{dx} + a_0\, y_n = -\, a_n \frac{d^n y_n}{dx^n}$$

$$\begin{bmatrix} y_1^{(n-1)} & y_1^{(n-2)} & \cdots & y_1' & y_1 \\ y_2^{(n-1)} & y_2^{(n-2)} & \cdots & y_2' & y_2 \\ & & \vdots & & \\ y_n^{(n-1)} & y_n^{(n-2)} & \cdots & y_n' & y_n \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} -a_n y_1^{(n)} \\ -a_n y_2^{(n)} \\ \vdots \\ -a_n y_n^{(n)} \end{bmatrix}$$

- Remark:

- Given $y_1(x),\ y_2(x), \cdots,\ y_n(x)$ be n solutions of $L(y) = 0$ on I

$W(f_1, f_2, \cdots, f_n)$ is:

either

IFF $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is

or

IFF $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is

- $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is a

- IF (1) $y_i(x),\ i = 1, 2, \cdots, n$ are

(2) $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is

- There exists a fundamental set of solutions for the homogeneous linear nth-order DE on I

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = 0$$

(1) $a_i(x)$, $i = 0, 1, \cdots, n$, are continuous on I

(2) $a_n(x) \neq 0$, $\forall x \in I$

- $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$:

  a fundamental set of solutions for the DE on I

- The general solution is

$$y(x) = \quad y_1(x) + \quad y_2(x) + \cdots + \quad y_n(x)$$

4.1.2: Example 7

$$y'' - 9y = 0$$

- $\left\{ \qquad , \qquad \right\}$:

$$W(\qquad , \qquad) = \det \begin{bmatrix} & \\ & \end{bmatrix}$$

$\Rightarrow y(x) =$

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

$$g(x) \neq 0$$

the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where

(1) $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is

(2) $y_p(x)$

(3) $c_i,\ i = 1, 2, \cdots, n,$

4.1.3: Thm 4.1.7: Superposition Principle – Nonhomogeneous Eqns

- IF

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y =$$

has a

- THEN

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y =$$

has a

4.1.3: Example 11

$$y'' - 3y' + 4y = -16x^2 + 24x - 8$$

$$y'' - 3y' + 4y = 2e^{2x}$$

$$y'' - 3y' + 4y = 2xe^{x} - e^{x}$$

$$y'' - 3y' + 4y = (-16x^2 + 24x - 8) + (2e^{2x}) + (2xe^{2x} - e^{x})$$

- 2nd-order DE:

Solve:

$$a_2(x) \frac{d^2y(x)}{dx^2} + a_1(x) \frac{dy(x)}{dx} + a_0(x)\, y(x) = g(x)$$

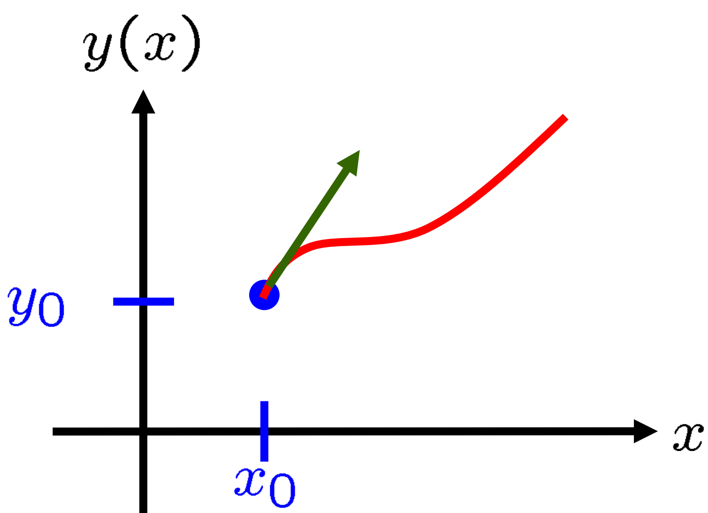
Subject to:

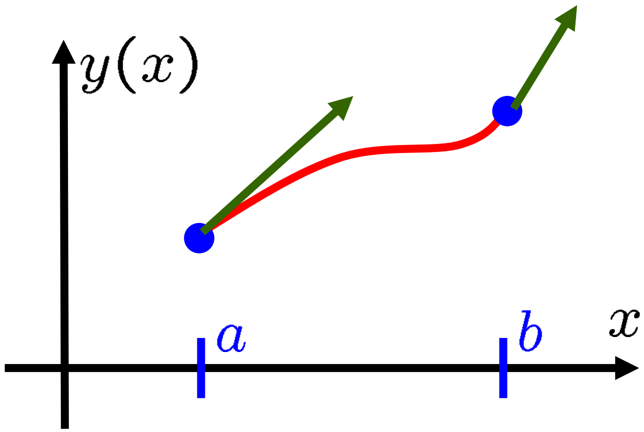
- Initial Condition

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

- Boundary Condition

$$\begin{cases} \alpha_1\, y(a) + \beta_1\, y'(a) = \gamma_1 \\ \alpha_2\, y(b) + \beta_2\, y'(b) = \gamma_2 \end{cases}$$





Summary - Thm 4.1.1: Existence of a Unique Solution

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ y^{(2)}(x_0) = y_2,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1}$

- Let $a_n(x),\ a_{n-1}(x),\ \cdots,\ a_1(x),\ a_0(x)$ & $g(x)$

  be continuous on I

- And $a_n(x) \neq 0,\ \forall x \in I,\quad x_0 \in I$

- Then IVP exsits a unique solution $y(x)$ on I

Summary - Thm 4.1.5: General Solutions – Homogeneous Eqns

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1}y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = 0$$

- $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$:

  a fundamental set of solutions for the DE on I

  (1) $y_i(x),\ i = 1, 2, \cdots, n$ are solutions

  (2) $\left\{ y_1(x),\ y_2(x), \cdots,\ y_n(x) \right\}$ is linearly independent

$$W(y_1, y_2, \cdots, y_n) \triangleq \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & \ddots & \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{bmatrix} \neq 0, \quad \forall x \in I$$

- The general solution:

$$y(x) = c_1\,y_1(x) + c_2\,y_2(x) + \cdots + c_n\,y_n(x)$$

Summary - Thm 4.1.6: General Solutions – Nonhomogeneous Eqns

$$a_n(x)\,\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y(x)}{dx^{n-1}} + \cdots + a_1(x)\,\frac{dy(x)}{dx} + a_0(x)\,y(x) = g(x)$$

$$g(x) \neq 0$$

- The general solution:

$$y(x) = c_1\,y_1(x) + c_2\,y_2(x) + \cdots + c_n\,y_n(x) + y_p(x)$$