Fall 2019

# 微分方程
# Differential Equations

## Unit 04.4
## Undetermined Coefficients – Superposition


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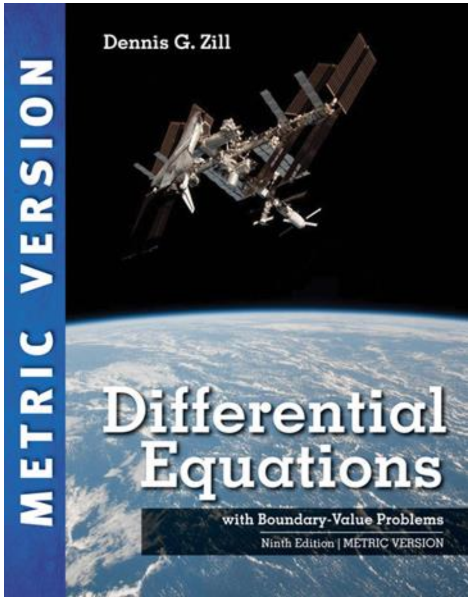


Figures and images used in these lecture notes are adopted from
**Differential Equations with Boundary-Value Problems**, 9th Ed., D.G. Zill, 2018 (Metric Version)

# Outline

# 4.4: Undetermined Coefficients – Superposition Approach

$$y'' + 4y' - 2y = 2x^2 - 6 + e^{-3x} - 4\cos(5x)$$

$\Rightarrow y = y_c + y_p$ $\Rightarrow$ AHE $\rightarrow y_c$

$\Rightarrow$ RHS $\rightarrow y_p$

$$y'' + 4y' - 2y = 6 \quad \Rightarrow y_p =$$

$$y'' + 4y' - 2y = 2x^2 \quad \Rightarrow y_p =$$

$$y'' + 4y' - 2y = e^{-3x} \quad \Rightarrow y_p =$$

$$y'' + 4y' - 2y = \cos(5x)$$

$$\Rightarrow y_p =$$

$$y'' + 4y' - 2y = \sin(5x)$$

$$y'' + 4y' - 2y = \ln x, \ \frac{1}{x}, \ \tan(x), \ \sin^{-1}(x), \ldots$$

# 4.4: Example 1

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p$$

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y = g(x)$$

$$\Rightarrow y =$$

$$y'' - y' + y = 2\sin(3x)$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' - y_p' + y_p =$$

# 4.4: Example 3

$$y'' - 2y' - 3y = 6e^{2x}$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' - y_p' + y_p =$$

# 4.4: Example 3

$$y'' - 2y' - 3y = 6xe^{2x}$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' - y_p' + y_p =$$

# 4.4: Example 4: But,

$$y'' - 5y' + 4y = 8e^x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \rightarrow y_c$$

$$\Rightarrow \text{RHS} \rightarrow y_p =$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' - y_p' + y_p =$$

# 4.4: Example 4: So,

$$y'' - 5y' + 4y = 8e^x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow \text{AHE} \to y_c$$

$$\Rightarrow \text{RHS} \to y_p =$$

$$y_p' =$$

$$y_p'' =$$

$$y_p'' - y_p' + y_p =$$

- Case I: Candidate particular solution is NOT a complementary function

**TABLE 4.1** Trial Particular Solutions

| $g(x)$ | Form of $y_p$ |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x + 7$ | $Ax + B$ |
| **3.** $3x^2 - 2$ | $Ax^2 + Bx + C$ |
| **4.** $x^3 - x + 1$ | $Ax^3 + Bx^2 + Cx + E$ |
| **5.** $\sin 4x$ | $A \cos 4x + B \sin 4x$ |
| **6.** $\cos 4x$ | $A \cos 4x + B \sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x - 2)e^{5x}$ | $(Ax + B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2 + Bx + C)e^{5x}$ |
| **10.** $e^{3x} \sin 4x$ | $Ae^{3x} \cos 4x + Be^{3x} \sin 4x$ |
| **11.** $5x^2 \sin 4x$ | $(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$ |
| **12.** $xe^{3x} \cos 4x$ | $(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$ |

- Case II: Candidate particular solution is ALSO a complementary function

$$y'' - 2\,y' + y = e^{x}$$

- $y_c$

- $y_p$

# 4.4: IVP & BVP & Example 8

$$y'' + y = 4x + 10\sin x$$

$$y(\pi) = 0$$
$$y'(\pi) = 2$$

- $y_c$

- $y_p$

$$y'' + 4\,y' - 2\,y = 2x^2 + 6 + 10\sin 3x - 3e^{-2x}$$

$$\Rightarrow y = y_c + y_p$$

4.4: $\Rightarrow \quad y_p = (Ax^2+Bx+C) + (E\cos 3x+F\sin 3x) + (Ge^{-2x})$

4.5: $\Rightarrow \quad P(D)\left(y'' + 4\,y' - 2\,y\right)$

$$= P(D)\left(2x^2 + 6 + 10\sin 3x - 3e^{-2x}\right) = 0$$

$e.g., \Rightarrow \quad y^{(4)} - 9\,y^{(3)} + 7\,y'' - 6\,y' + 12\,y = 0$

$$\Rightarrow y = y_c + y_p$$