Fall 2019

微分方程
Differential Equations

Unit 04.7
Cauchy-Euler Equations

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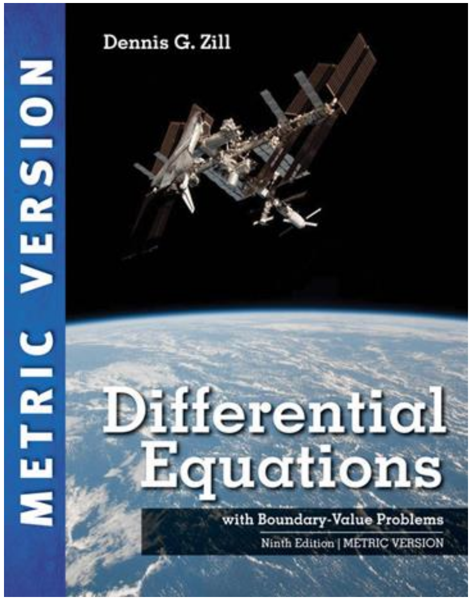
$$ax^2y'' + bxy' + cy = 0$$

$$y(x) = x^m$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

$$a_i \in \mathbb{R}, \quad I = (0, \infty)$$

known as a Cauchy-Euler Equation (Equidimensional Eqn)

Augustin-Louis Cauchy (1789-1857)
Leonhard Euler (1707-1783)

- Consider a 2nd-order DE:

$$a x^2 y'' + b x y' + c y = 0$$

- Assume $y(x) =$

$y'(x) =$

$y''(x) =$

- Case I: distinct real roots: m_1, m_2

- Case II: repeated real roots: $m_1 = m_2$

$$\begin{cases} y_1(x) = \\ y_2(x) = \end{cases}$$

$$y_2'(x) =$$

$$y_2''(x) =$$

- Case III: conjugate complex roots

$$\begin{cases} m_1 = \\ \\ m_2 = \end{cases}$$

$$\Rightarrow \quad x^{m_1} =$$

$$\Rightarrow \quad x^{m_2} =$$

$$\Rightarrow \quad y(x) =$$

4.7: Another Point of View

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0$$

- Let $x =$

$y(x) \longrightarrow$

- auxiliary eqn:

- Case I: m_1, m_2 distinct real roots

- Case II: $m_1 = m_2$

- Case III: $m_{1,2} = \alpha \pm i\beta$

4.7: Example 5

$$x^2\, y'' - 3\, x\, y' + 3\, y = 2\, x^4\, e^x$$

- auxiliary eqn. $y(x) =$

$\Rightarrow\ y_c =$

$\Rightarrow\ y_p =$

4.7: Another Form of Cauchy-Euler Equation

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0$$

$$a\, (x - x_0)^2\, y'' + b\, (x - x_0)\, y' + c\, y = 0$$

- Consider a 2nd-order DE: $a\,x^2\,y'' + b\,x\,y' + c\,y = 0$

- Assume

$$y(x) = x^m$$
$$y'(x) = m\,x^{m-1}$$
$$y''(x) = m\,(m-1)\,x^{m-2}$$

$$\Rightarrow a\,m(m-1) + b\,m + c = 0$$

$$\Rightarrow a\,m^2 + (b-a)\,m + c = 0$$

- Case I: distinct real roots: m_1, m_2

$$y(x) = c_1\,x^{m_1} + c_2\,x^{m_2}$$

- Case II: repeated real roots: $m_1 = m_2$

$$y(x) = c_1\,x^{m_1} + c_2\ln x\,x^{m_1}$$

- Case III: conjugate complex roots $m_{1,2} = a \pm ib$

$$y(x) = x^a\left(c_1\cos(b\ln x) + c_2\sin(b\ln x)\right)$$