Fall 2019

微分方程
Differential Equations

Unit 07.2
Inverse Transforms
and Transforms of Derivatives

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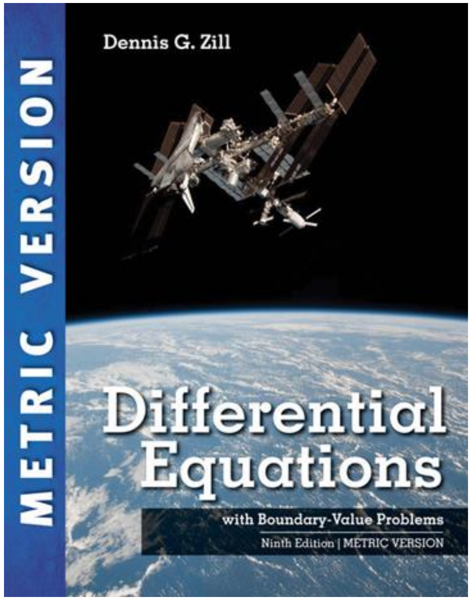
$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

- Given $F(s)$ $\Rightarrow$ Find $f(t)$

$$\Rightarrow \mathcal{L}^{-1}\left\{ \qquad \right\} =$$

7.2: Theorem 7.2.1: Some Inverse Transforms

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

(b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1, 2, 3, \cdots$

(c) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(d) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$ (e) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

(f) $\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$ (g) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

See Appendix III for a list of Laplace transforms

7.2: Example

$$\mathcal{L}^{-1}\left\{\frac{3s-2}{s^3(s^2+4)}\right\} =$$

7.2: Theorem 7.2.2: Transform of a Derivative

- If $f, f', \cdots, f^{(n-1)}$ are on $[0, \infty)$

  are of

- and, if $f^{(n)}$ is on $[0, \infty)$

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

- then $\mathcal{L}\left\{f^{(n)}(t)\right\} =$

$$F(s)$$

$$- \quad f(0)$$

$$- \quad f'(0) \quad - \quad \cdots$$

$$- \quad f^{(n-1)}(0)$$

7.2: Theorem 7.2.2: Transform of a Derivative

$$\mathcal{L}\left\{ f'(t) \right\} \quad =$$

$$\mathcal{L}\left\{ f^{(2)}(t) \right\} \quad =$$

7.2: Example 5: Solving a 2nd-Order IVP

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

7.2: Example 5:

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^{\infty} e^{-st} f(t) \, dt = F(s)$$

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) \, ds = f(t)$$

$$(a) \quad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$(b) \quad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1, 2, 3, \cdots$$

$$(c) \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$(d) \quad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt \qquad (e) \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

$$(f) \quad \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt \qquad (g) \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$

Summary - 7.2: Inverse Laplace T. and T. of Derivatives

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{f''(t)\right\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

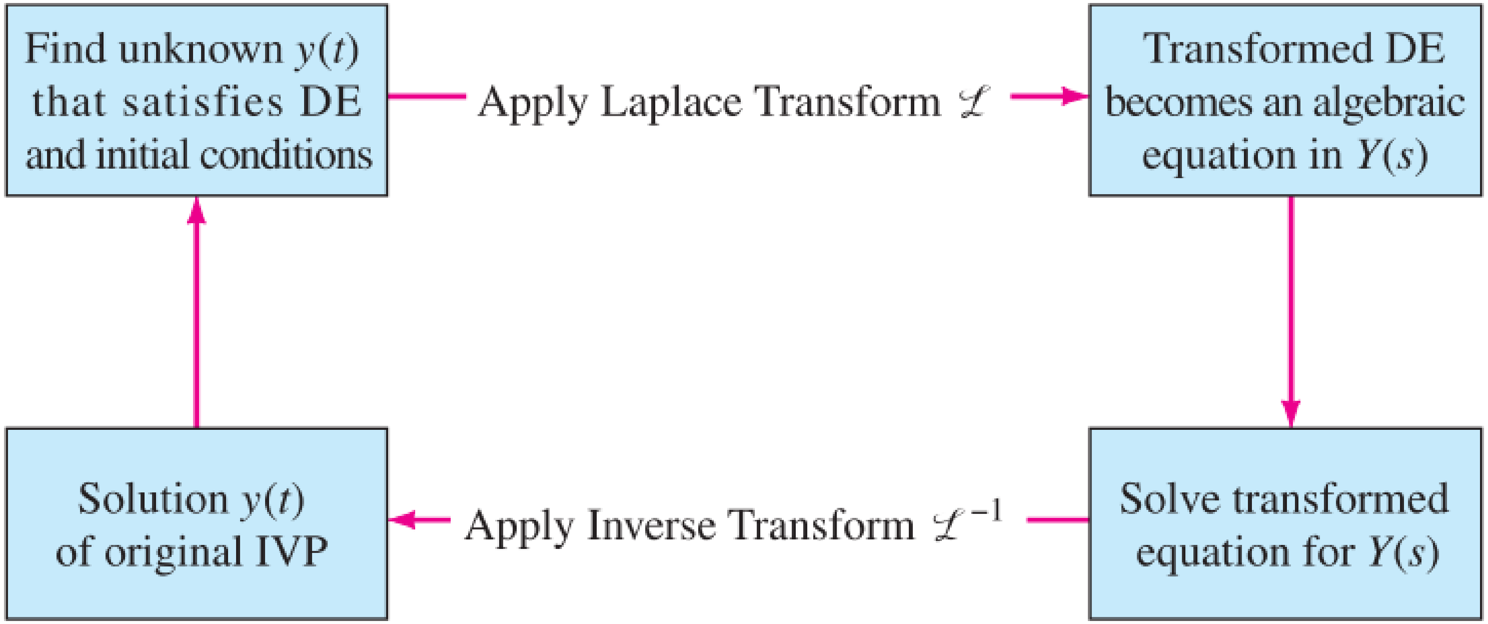
$$y'' - 3y' + 2y = e^{-4t}$$

$$\mathcal{L}\left\{y'' - 3y' + 2y\right\} = \left(s^2 - 3s + 2\right) Y(s) - sy(0) - y'(0) + 3y(0)$$

$$\mathcal{L}\left\{e^{-4t}\right\} = \frac{1}{s+4}$$

$$Y(s) = \ldots \longrightarrow y(t) = \ldots$$

Summary - 7.2: Inverse Laplace T. and T. of Derivatives



7.2: Example 5: Solving a 2nd-Order IVP

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 4.2:

$$y_1(t) = e^t$$

$$y(t) = u(t)\, y_1(t) = u\, e^t$$

$$y'(t) = u'\, y_1 + u\, y_1' = (u' + u)\, e^t$$

$$y''(t) = u''\, y_1 + 2\, u'\, y_1' + u\, y_1'' = (u'' + 2u' + u)\, e^t$$

$$\Rightarrow (u'' + 2u' + u)\, e^t \quad -3(u' + u)\, e^t \quad +2u\, e^t = 0$$

$$\Rightarrow (u'' - u')\, e^t = 0$$

$$\Rightarrow (u'' - u') = 0$$

$$\rightarrow \text{ let } w = u' \qquad \Rightarrow e^{-t}\, w = c_1 \qquad \Rightarrow u = c_1\, e^t + c_2$$

$$\Rightarrow (w' - w) = 0 \qquad \Rightarrow w = c_1\, e^t \qquad \Rightarrow y = u\, y_1$$

$$\Rightarrow \text{I.F.} = e^{-t} \qquad \Rightarrow u' = w = c_1\, e^t \qquad = c_1\, e^{2t} + c_2\, e^t$$

$$\Rightarrow \frac{d}{dt}\left[e^{-t}\, w\right] = 0 \qquad \Rightarrow u = \int c_1\, e^t dt$$

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 4.3 & 4.4:

$$y(t) = e^{mt} \qquad \Rightarrow \quad y_p(t) = Ae^{-4t}$$

$$y'(t) = me^{mt} \qquad y_p'(t) = A(-4)e^{-4t}$$

$$y''(t) = m^2 e^{mt} \qquad y_p''(t) = A(-4)^2 e^{-4t}$$

$$y'' - 3y' + 2y = 0 \qquad y_p'' - 3y_p' + 2y_p = e^{-4t}$$

$$\Rightarrow \quad m^2 - 3m + 2 = 0 \qquad (16A + 12A + 2A)e^{-4t} = e^{-4t}$$

$$\Rightarrow \quad m_{1,2} = 1, 2 \qquad \Rightarrow \quad A = \frac{1}{30}$$

$$\Rightarrow \quad y_c(t) = c_1 e^{1t} + c_2 e^{2t} \qquad \Rightarrow \quad y_p(t) = \frac{1}{30} e^{-4t}$$

$$\Rightarrow \quad y(t) = y_c(t) + y_p(t) = c_1 e^{1t} + c_2 e^{2t} + \frac{1}{30} e^{-4t}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases} \qquad = \frac{-16}{5} e^{1t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

7.2: Example 5: Solving a 2nd-Order IVP

$$y'' - 3y' + 2y = e^{-4t} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 5 \end{cases}$$

- Method in 7.2:

$$\mathcal{L}\left\{y'' - 3y' + 2y\right\} = \mathcal{L}\left\{e^{-4t}\right\}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) - y(0)(s-3) - y'(0) = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2)Y(s) = y(0)(s-3) + y'(0) + \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2 - 3s + 2)}\Big(y(0)(s-3) + y'(0)\Big)$$

- zero-input response

$$+ \frac{1}{(s^2 - 3s + 2)}\left(\frac{1}{s+4}\right)$$

- zero-state response

$$= \left(\frac{-16}{5}\right)\frac{1}{s-1} + \left(\frac{25}{6}\right)\frac{1}{s-2} + \left(\frac{1}{30}\right)\frac{1}{s+4}$$

$$\Rightarrow y(t) = \left(\frac{-16}{5}\right)e^{1t} + \left(\frac{25}{6}\right)e^{2t} + \left(\frac{1}{30}\right)e^{-4t}$$