Fall 2019

微分方程
Differential Equations

Unit 07.3
Operational Properties I

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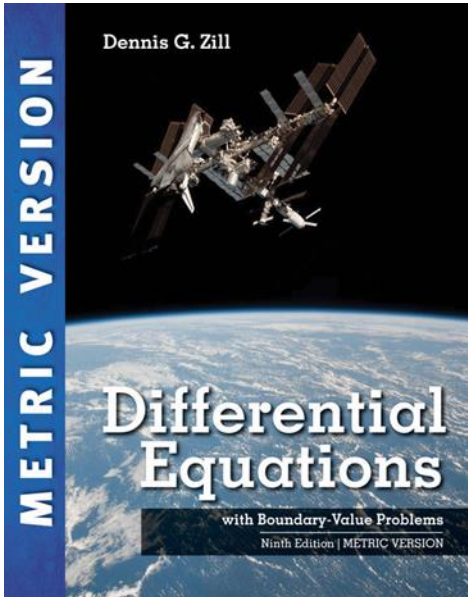
NTU-EE

Sep19 – Jan20

$$\mathcal{L}\left\{e^{at}\, f(t)\right\} = F(s-a)$$

$$\mathcal{L}\left\{f(t-a)\,\mathcal{U}(t-a)\right\} = e^{-as}\, F(s)$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^{\infty} e^{-st} f(t) \, dt = F(s)$$

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$$

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$$f(t \qquad) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$$

- Theorem 7.3.1: First Translation Theorem

IF $\mathcal{L}\left\{ f(t) \right\} = F(s)$ AND, $a \in \mathbb{R}$

THEN $\mathcal{L}\left\{ \qquad f(t) \right\} =$

- Proof:

7.3.1: Examples

$$\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}$$

$$\Rightarrow \mathcal{L}\left\{e^{at}t^3\right\} =$$

$$\mathcal{L}\left\{\cos 4t\right\} = \frac{s}{s^2+4^2}$$

$$\Rightarrow \mathcal{L}\left\{e^{at}\cos 4t\right\} =$$

$$\mathcal{L}\left\{\sin 4t\right\} = \frac{4}{s^2+4^2}$$

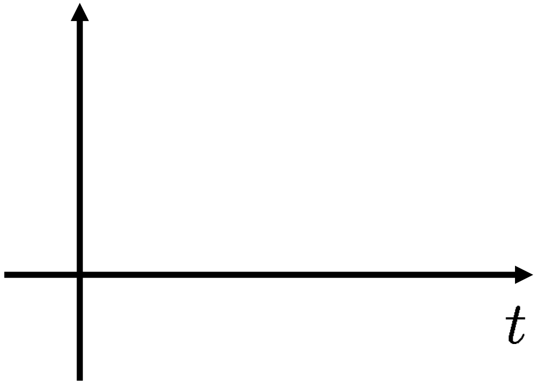
$$\Rightarrow \mathcal{L}\left\{e^{at}\sin 4t\right\} =$$

7.3.1: Examples

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+11}\right\} =$$

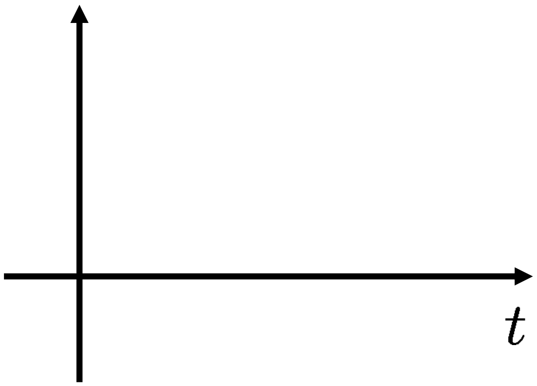
- Definition 7.3.1: Unit Step Function

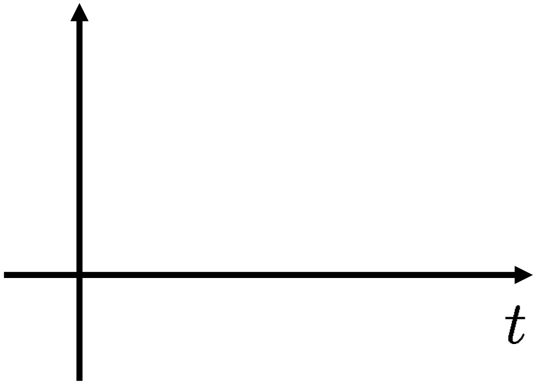
$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t \end{cases}$$



$$\Rightarrow \mathcal{U}(t-a) - \mathcal{U}(t-b)$$

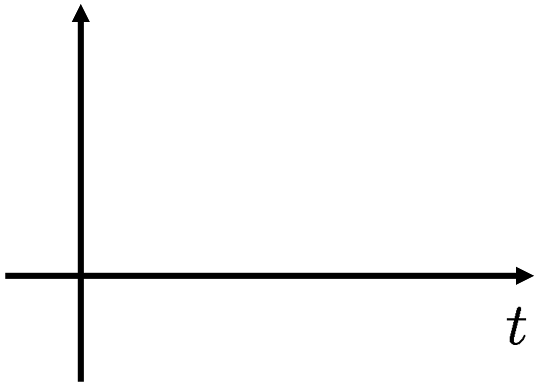
$$a < b$$





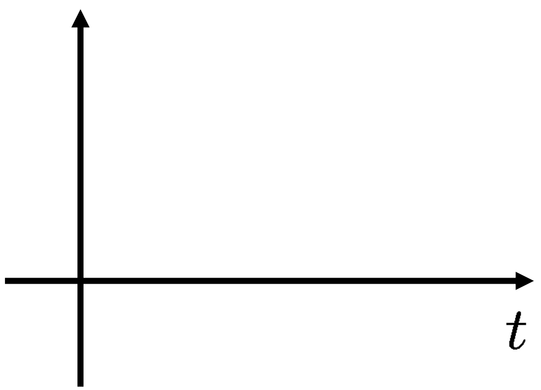
7.3.2: Translation on the t-Axis

- $f(t) \; = \; t, \qquad t \geq 0$



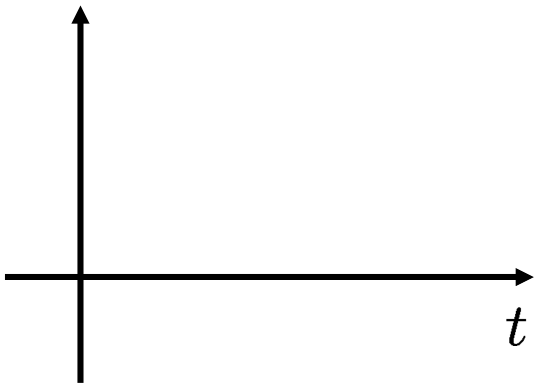
- $g(t) \; = \; f(t)\, \mathcal{U}(t-a) \; =$

$=$



- $h(t) \; = \; f(t-a)\, \mathcal{U}(t-a) \; =$

$=$



- Theorem 7.3.2: 2nd Translation Theorem

IF $\mathcal{L}\left\{ f(t) \right\} = F(s)$ AND, $a > 0$

THEN $\mathcal{L}\left\{ f(t \qquad) \qquad \right\} = \qquad F(s)$

- Proof:

7.3.2: Translation on the t-Axis

- $\mathcal{L}\left\{\mathcal{U}(t-a)\right\} =$

- $\mathcal{L}\left\{g(t)\,\mathcal{U}(t-a)\right\} =$

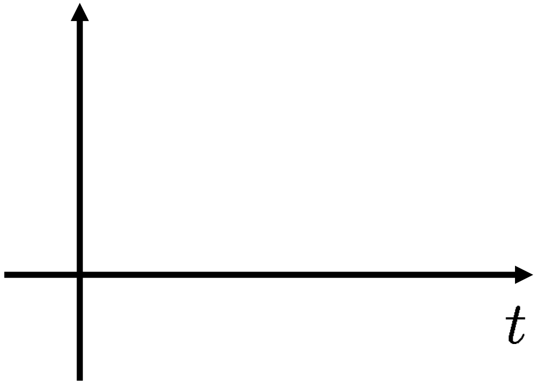
- $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} =$

7.3.2: Examples

$$\mathcal{L}\left\{(t-2)^3\,\mathcal{U}(t-2)\right\} =$$

7.3.2: Examples

$$\mathcal{L}\left\{2 - 3\,\mathcal{U}(t-2) + \mathcal{U}(t-3)\right\} =$$



7.3.2: Examples

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s^2+9}\right\} =$$

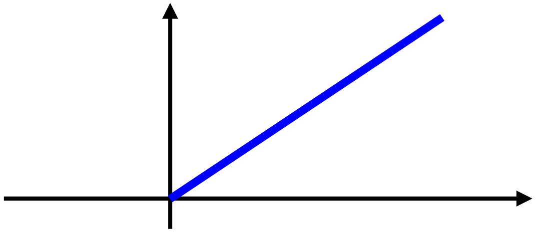
7.3.2: Examples

$$\mathcal{L}\left\{(2t - 3)\,\mathcal{U}(t-1)\right\} =$$

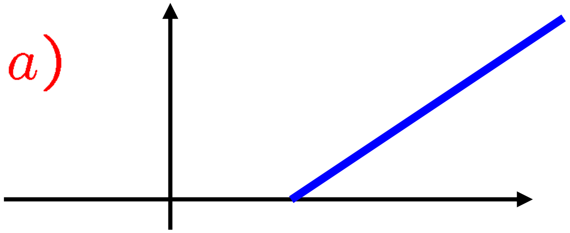
7.3.2: More Comparisons

$f(t) = t,$

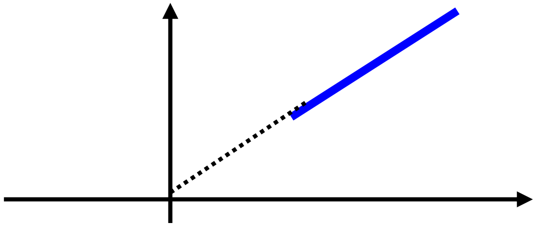
$t \geq 0$



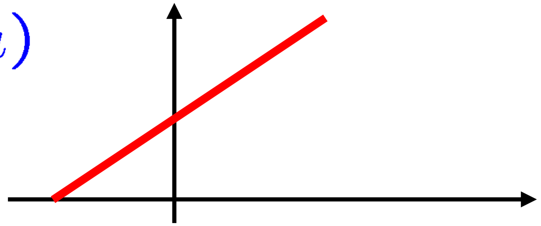
$f(t-a)\mathcal{U}(t-a)$



$f(t)\mathcal{U}(t-a)$

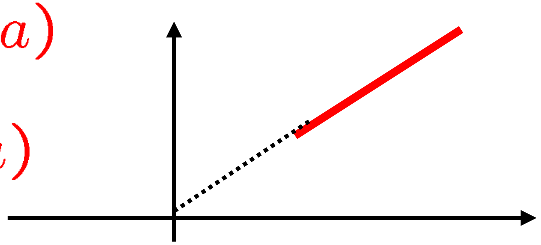


$g(t) = f(t+a)$



$g(t-a)\mathcal{U}(t-a)$

$= f(t)\mathcal{U}(t-a)$



Summary - 7.3: Translation on the s-Axis and the t-Axis

$$\mathcal{L}\left\{ \cos(4t) \right\} = \frac{s}{s^2+4^2}$$

$$s \longrightarrow (s-3)$$

$$\mathcal{L}\left\{ e^{3t} \cos(4t) \right\} = \frac{(s-3)}{(s-3)^2+4^2}$$

$$t \longrightarrow (t-\frac{\pi}{2})$$

$$\mathcal{L}\left\{ \cos(4(t-\frac{\pi}{2}))\, \mathcal{U}(t-\frac{\pi}{2}) \right\} = \frac{s}{s^2+4^2} \left(e^{-\frac{\pi}{2}s}\right)$$

$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$\mathcal{L}\left\{f(t-a)\, \mathcal{U}(t-a)\right\} = e^{-as} F(s)$$