

Fall 2019

# 微分方程 Differential Equations

## Unit 07.3 Operational Properties I

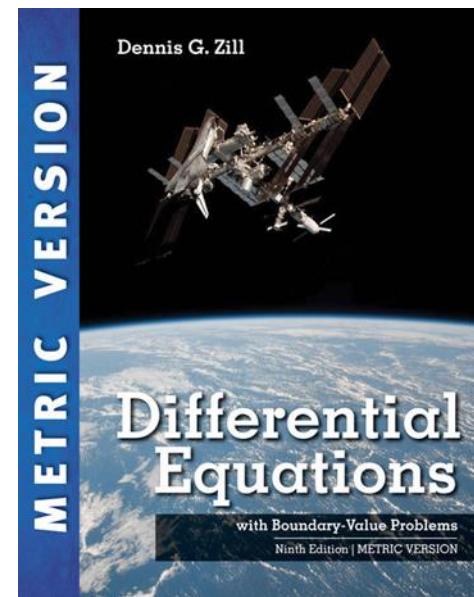
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NTU-EE

$$\mathcal{L} \left\{ e^{\textcolor{red}{at}} f(t) \right\} = F(\textcolor{red}{s} - a) \quad \text{Sep19 – Jan20}$$

$$\mathcal{L} \left\{ f(\textcolor{red}{t} - a) \mathcal{U}(\textcolor{red}{t} - a) \right\} = e^{-\textcolor{red}{as}} F(s)$$

Figures and images used in these lecture notes are adopted from  
**Differential Equations with Boundary-Value Problems**, 9th Ed., D.G. Zill, 2018 (Metric Version)



- 7.1: Definition of Laplace Transform
- 7.2: Inverse Transforms and Transforms of Derivatives
  - 7.2.1: Inverse Transforms
  - 7.2.2: Transforms of Derivatives
- **7.3: Operational Properties I**
  - **7.3.1: Translation on the s-Axis**
  - **7.3.2: Translation on the t-Axis**
- 7.4: Operational Properties II
  - 7.4.1: Derivatives of a Transform
  - 7.4.2: Transforms of Integrals
  - 7.4.3: Transform of a Periodic Function
- 7.5: The Dirac Delta Function
- 7.6: Systems of Linear Differential Equations

$$\mathcal{L} \left\{ f(t) \right\} \triangleq \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\mathcal{L} \left\{ f(t) \right\} = F(s)$$

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

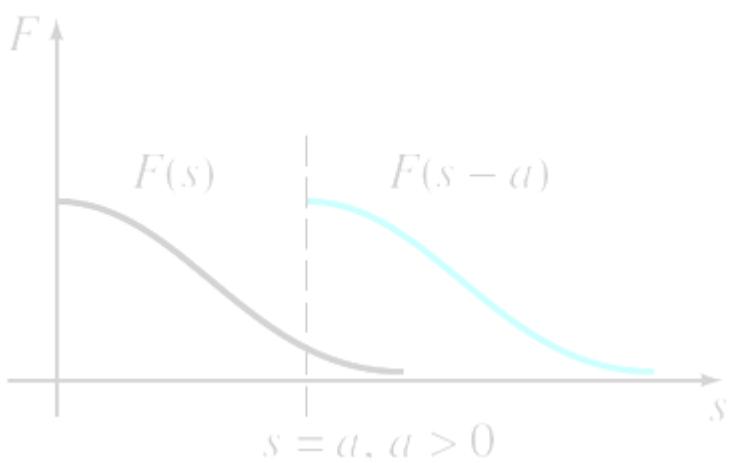
$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

- Theorem 7.3.1: First Translation Theorem

IF  $\mathcal{L}\{f(t)\} = F(s)$  AND,  $a \in \mathbb{R}$

THEN  $\mathcal{L}\{f(t)\} =$

- Proof:



$$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

$$\Rightarrow \mathcal{L}\{e^{at}t^3\} =$$

$$\mathcal{L}\{\cos 4t\} = \frac{s}{s^2 + 4^2}$$

$$\Rightarrow \mathcal{L}\{e^{at}\cos 4t\} =$$

$$\mathcal{L}\{\sin 4t\} = \frac{4}{s^2 + 4^2}$$

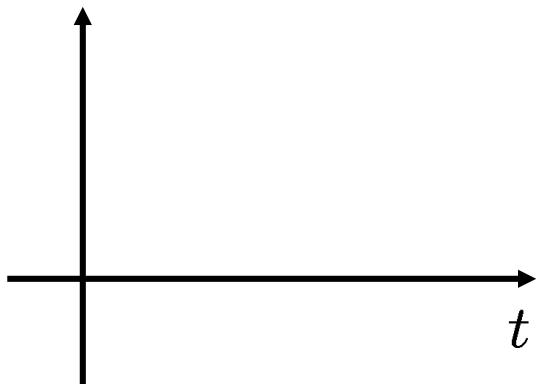
$$\Rightarrow \mathcal{L}\{e^{at}\sin 4t\} =$$

## 7.3.1: Examples

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 6s + 11} \right\} =$$

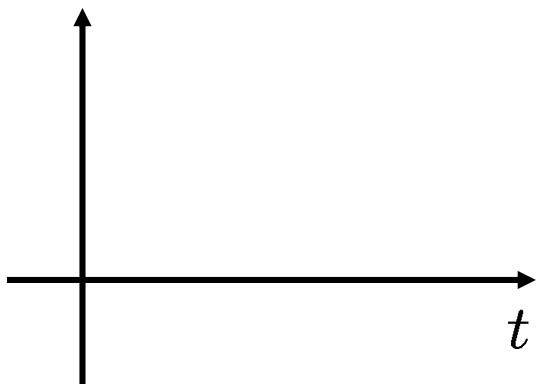
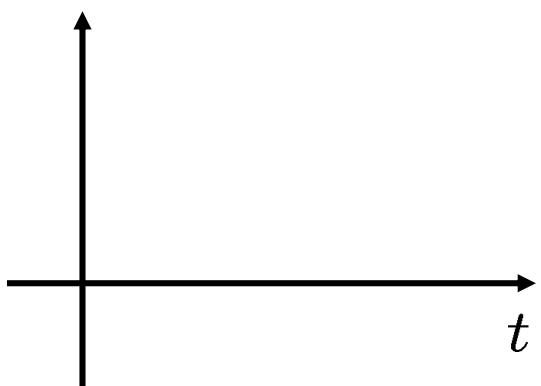
- Definition 7.3.1: Unit Step Function

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & a \leq t \end{cases}$$



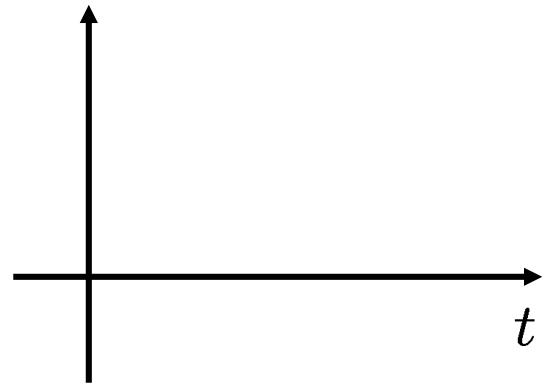
$$\Rightarrow \mathcal{U}(t - a) - \mathcal{U}(t - b)$$

$$a < b$$



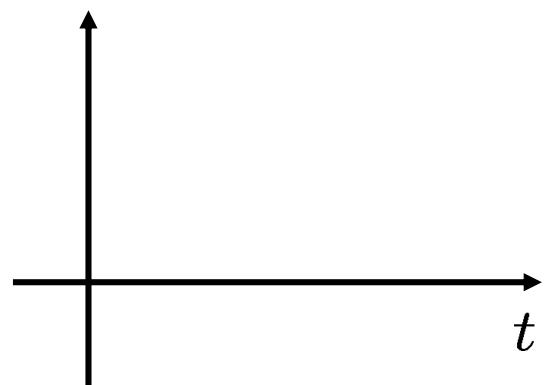
### 7.3.2: Translation on the t-Axis

- $f(t) = t, \quad t \geq 0$



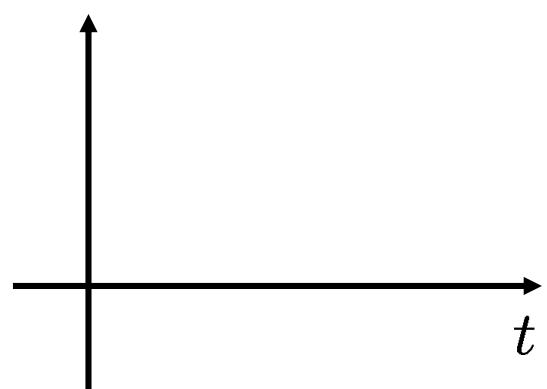
- $g(t) = f(t) \mathcal{U}(t - a) =$

=



- $h(t) = f(t - a) \mathcal{U}(t - a) =$

=



- Theorem 7.3.2: 2nd Translation Theorem

IF  $\mathcal{L} \{ f(t) \} = F(s)$  AND,  $a > 0$

THEN  $\mathcal{L} \{ f(t-a) \} = F(s)$

- Proof:

$$\bullet \mathcal{L} \left\{ \mathcal{U}(t-a) \right\} =$$

$$\bullet \mathcal{L} \left\{ g(t) \mathcal{U}(t-a) \right\} =$$

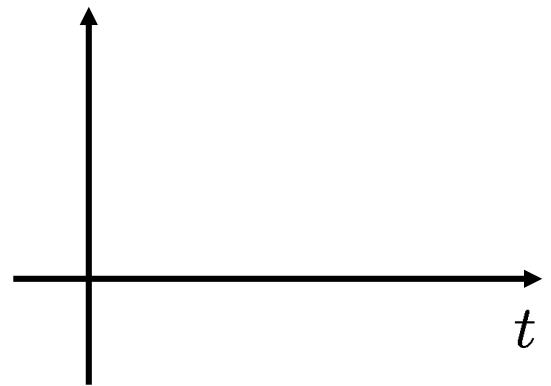
$$\bullet \mathcal{L}^{-1} \left\{ e^{-as} F(s) \right\} =$$

## 7.3.2: Examples

$$\mathcal{L} \left\{ (t-2)^3 \mathcal{U}(t-2) \right\} =$$

## 7.3.2: Examples

$$\mathcal{L} \left\{ 2 - 3U(t-2) + U(t-3) \right\} =$$



## 7.3.2: Examples

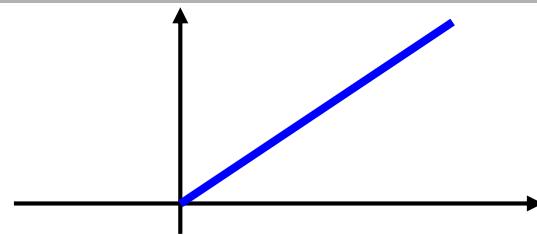
$$\mathcal{L}^{-1} \left\{ \frac{e^{-\frac{\pi}{2}s}}{s^2 + 9} \right\} =$$

## 7.3.2: Examples

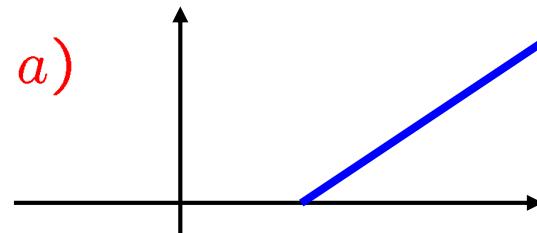
$$\mathcal{L} \left\{ (2t - 3) \mathcal{U}(t - 1) \right\} =$$

### 7.3.2: More Comparisons

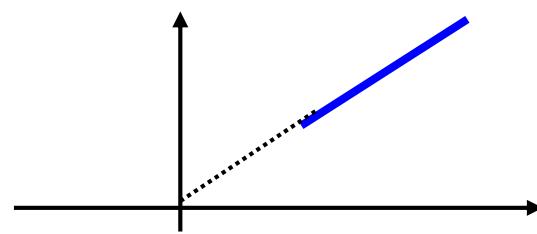
$$f(t) = t, \\ t \geq 0$$



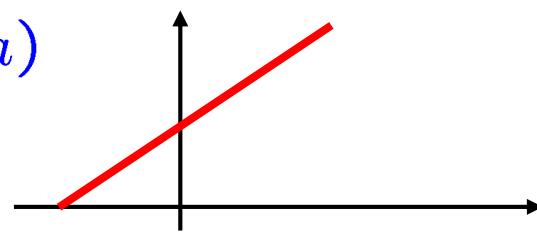
$$f(t - a)\mathcal{U}(t - a)$$



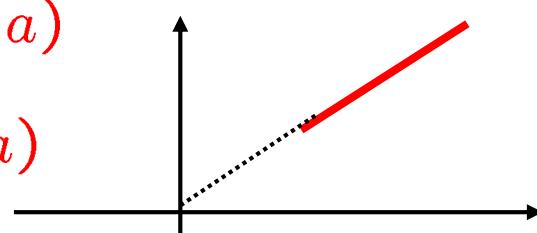
$$f(t)\mathcal{U}(t - a)$$



$$g(t) = f(t + a)$$



$$g(t - a)\mathcal{U}(t - a)$$



$$= f(t)\mathcal{U}(t - a)$$

$$\mathcal{L} \left\{ \cos(4t) \right\} = \frac{s}{s^2 + 4^2}$$

$$s \rightarrow (s - 3)$$

$$\mathcal{L} \left\{ e^{3t} \cos(4t) \right\} = \frac{(s - 3)}{(s - 3)^2 + 4^2}$$

$$t \rightarrow (t - \frac{\pi}{2})$$

$$\mathcal{L} \left\{ \cos(4(t - \frac{\pi}{2})) \mathcal{U}(t - \frac{\pi}{2}) \right\} = \frac{s}{s^2 + 4^2} \left( e^{-\frac{\pi}{2}s} \right)$$

$$\mathcal{L} \left\{ f(t) \right\} \triangleq \int_0^\infty e^{-st} f(t) dt = F(s)$$

$$\mathcal{L} \left\{ f(t) \right\} = F(s)$$

$$\mathcal{L} \left\{ e^{at} f(t) \right\} = F(s - a)$$

$$\mathcal{L} \left\{ f(t - a) \mathcal{U}(t - a) \right\} = e^{-as} F(s)$$