Fall 2019

微分方程
Differential Equations

Unit 07.4
Operational Properties II

$$\mathcal{L}\left\{t^n\, f(t)\right\}$$

$$\mathcal{L}\left\{f(t) * g(t)\right\}$$

$$\mathcal{L}\left\{f(t+T)\right\}$$

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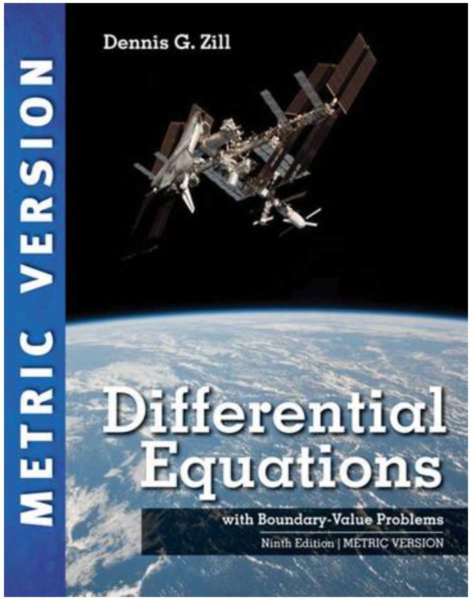
Sep19 – Jan20



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

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- Theorem 7.4.1: Derivatives of Transforms

IF $\mathcal{L}\left\{f(t)\right\} = F(s)$

THEN $\mathcal{L}\left\{ \qquad f(t)\right\} = \qquad F(s)$

- Proof:

7.4: Theorem 7.4.1

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7.4: Example

$$\mathcal{L}\left\{ t \ e^{-t} \ \cos t \right\} =$$

- method 1

$$\mathcal{L}\left\{ \cos t \right\} =$$

$$\mathcal{L}\left\{ e^{-t} \ \cos t \right\} =$$

$$\mathcal{L}\left\{ t \ e^{-t} \ \cos t \right\} =$$

7.4: Example

$$\mathcal{L}\left\{ t\ e^{-t}\ \cos t \right\} =$$

- <u>method 2</u>

$$\mathcal{L}\left\{ \cos t \right\} =$$

$$\mathcal{L}\left\{ t\ \cos t \right\} =$$

$$\mathcal{L}\left\{ e^{-t}\ t\ \cos t \right\} =$$

7.4.2: Transforms of Integrals: Convolution

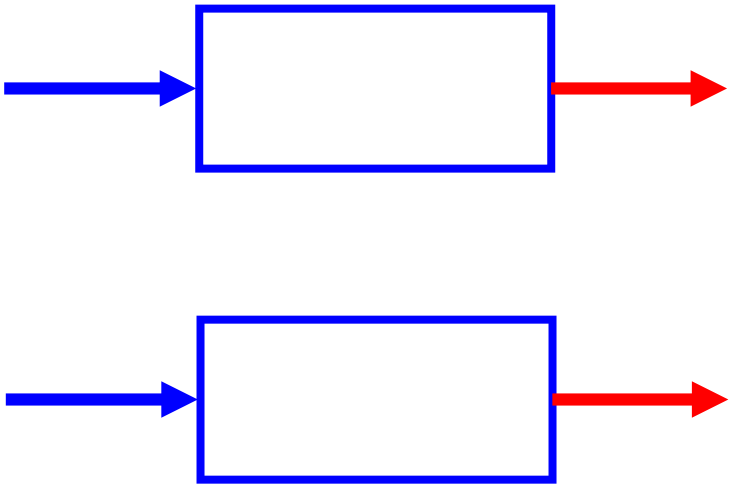
- $f(t)$ and $g(t)$:

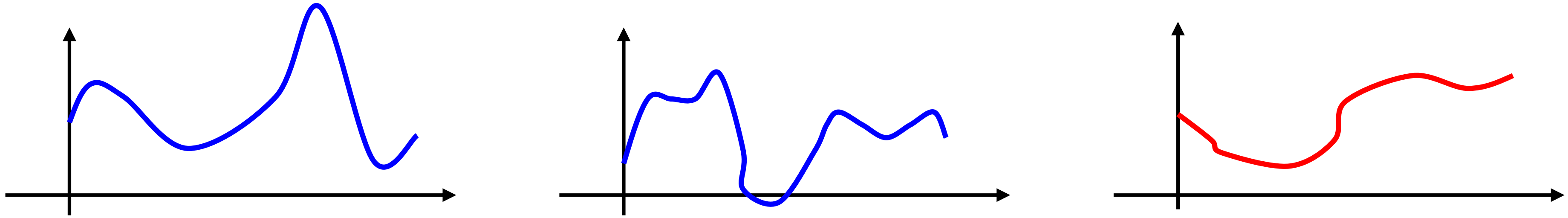
$$\Rightarrow f(t) \ast g(t) =$$

7.4.2: Transforms of Integrals: Convolution

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = h(t)$$

$$= \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$





- Theorem 7.4.2: Convolution Theorem

IF $f(t)$ and $g(t)$:

THEN $\mathcal{L}\left\{ f(t) * g(t) \right\} =$

- Proof:

7.4.2: Transforms of Integrals: Convolution Theorem

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7.4.2: Transform of an Integral

- $f(t) * \mathcal{U}(t) \quad = \; ?$

7.4.2: Summary

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$$f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad F(s)$$

$$\frac{d}{dt}\, f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

$$\frac{d^n}{dt^n}\, f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

$$\int_0^t f(\tau)\, d\tau \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

$$\int_0^t \left(\int_0^{\tau_{n-1}} \left(\cdots \left(\int_0^{\tau_1} f(\tau)\, d\tau \right) d\tau_1 \right) \cdots \right) d\tau_{n-1} \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

$$-t\; f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

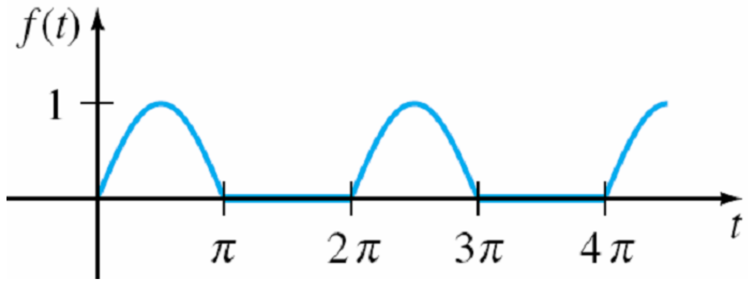
$$t^2\; f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

$$(-1)^n\, t^n\; f(t) \quad \overset{\mathcal{L}}{\longleftrightarrow}$$

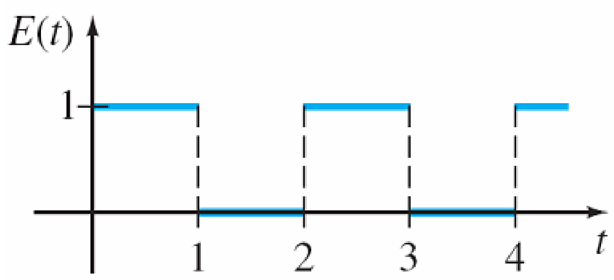
- Definition: Periodic Function

$f(t)$ is a periodic function with a period $T > 0$

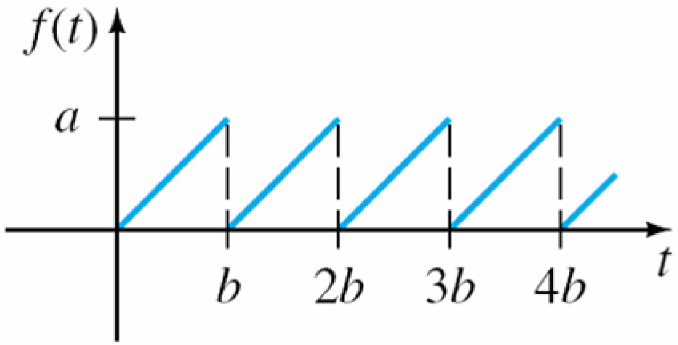
if $\quad f(t + T) =$



half-wave rectification of sin t



square wave



sawtooth

- Theorem 7.4.3:

IF $f(t)$: $\begin{cases} (1) \\ (2) \\ (3) \end{cases}$

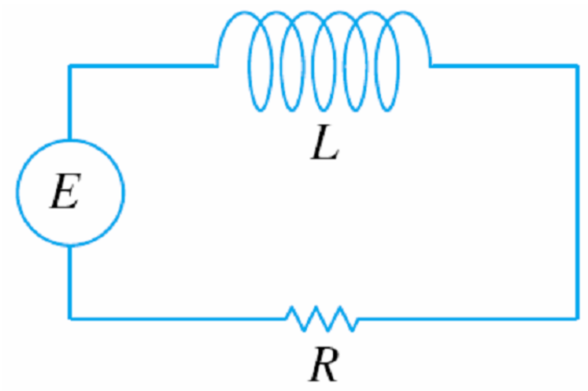
THEN $\mathcal{L}\left\{f(t)\right\} =$

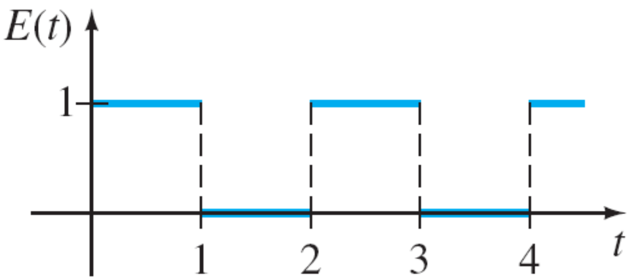
- Proof:

$$F(s) = \mathcal{L}\left\{f(t)\right\} =$$

7.4.3: Transform of a Periodic Function

7.4.3: Examples 8 & 9





$$L \frac{d}{dt} i(t) + R\, i(t) = \qquad E(t) \qquad i(0) = 0$$

$$\Rightarrow L\, s\, I(s) +$$

$$\Rightarrow \mathcal{L}\left\{ E(t) \right\} =$$

7.4.3: Examples 8 & 9

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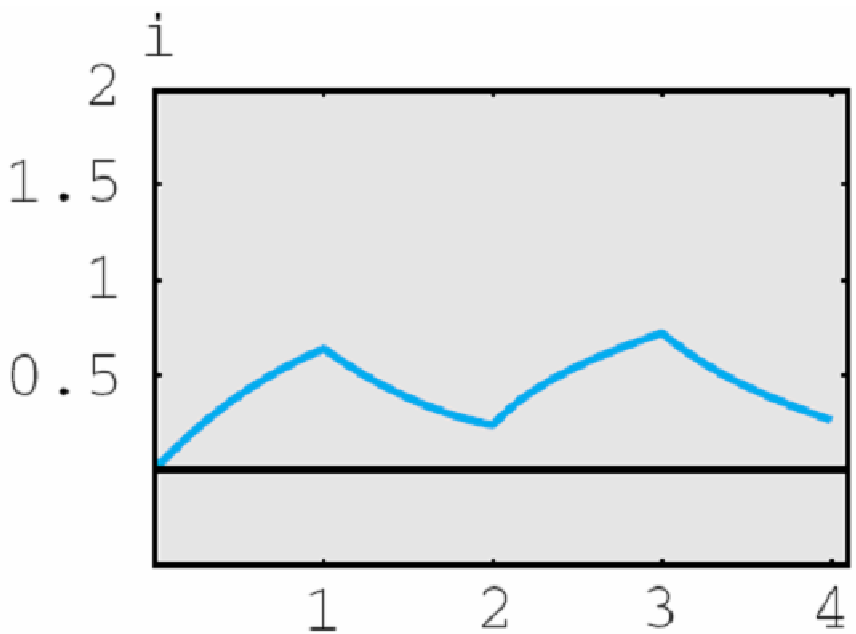
7.4.3: Examples 8 & 9

- IF $R = 1,\ L = 1,\ 0 \le t < 4$

$$i(t) = \begin{cases} 1 - e^{-(t)}, & 0 \le t < 1 \\ -e^{-(t)} + e^{-(t-1)}, & 1 \le t < 2 \\ 1 - e^{-(t)} + e^{-(t-1)} - e^{-(t-2)}, & 2 \le t < 3 \\ -e^{-(t)} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)}, & 3 \le t < 4 \end{cases}$$



Summary - 7.4: Derivatives of T., T. of Integrals & Periodic F.

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$$\mathcal{L}\left\{f(t)\right\} \triangleq \int_0^{\infty} e^{-st} f(t) \, dt = F(s)$$

$$\mathcal{L}\left\{f(t)\right\} = F(s)$$

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}\left\{f(t) * g(t)\right\} = \mathcal{L}\left\{f(t)\right\} \mathcal{L}\left\{g(t)\right\} = F(s) \, G(s)$$

$$f(t + T) = f(t)$$

$$\mathcal{L}\left\{f(t)\right\} = \frac{1}{1 - e^{sT}} \int_0^{T} e^{-st} f(t) \, dt$$