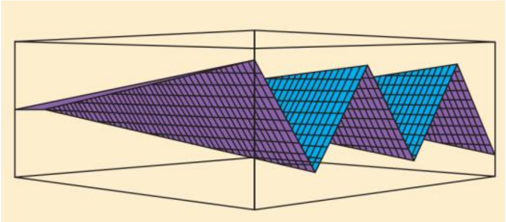
Fall 2019



微分方程
Differential Equations

Unit 12.1
Separable Partial Differential Equations

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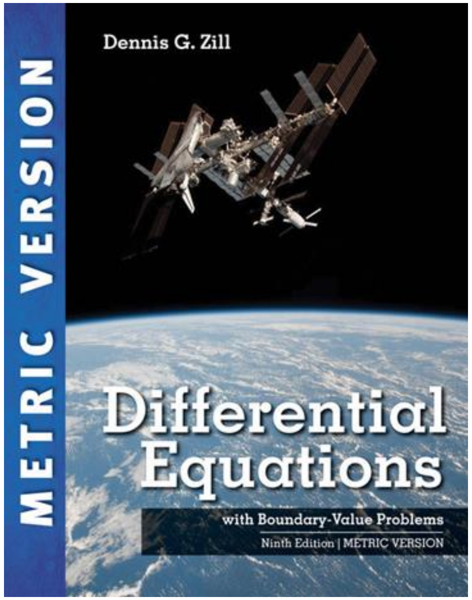
$$\frac{\partial^2 u}{\partial x^2} = 4\,\frac{\partial u}{\partial y}$$

$$u(x, y) = X(x)\,Y(y)$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

- **12.1: Separable Partial Differential Equations**
- 12.2: Classical PDEs and BVPs
- 12.3: Heat Equation
- 12.4: Wave Equation
- 12.5: Laplace's Equation
- 12.6: Nonhomogeneous BVPs
- 12.7: Orthogonal Series Expansions
- 12.8: Higher-Dimensional Problems

- Linear Second-Order PDE:

$$A(x,y)\frac{\partial^2 u(x,y)}{\partial x^2} + B(x,y)\frac{\partial^2 u(x,y)}{\partial x\, \partial y} + C(x,y)\frac{\partial^2 u(x,y)}{\partial y^2}$$

$$+ D(x,y)\frac{\partial u(x,y)}{\partial x} + E(x,y)\frac{\partial u(x,y)}{\partial y} + F(x,y)u(x,y)$$

$$= G(x,y) \qquad \begin{cases} G(x,y) = 0 & \text{Homogeneous} \\ G(x,y) \neq 0 & \text{Nonhomogeneous} \end{cases}$$

- Finding general solutions is very difficult
  Not all that useful in applications

- Finding particular solutions of
  some of more important linear PDE
  appearing in many applications

- Assume:

$$u(x,y) = X(x)\,Y(y)$$

- Then

$$\frac{\partial u}{\partial x} = X'\,Y \qquad \frac{\partial^2 u}{\partial x^2} = X''\,Y$$

$$\frac{\partial u}{\partial y} = X\,Y' \qquad \frac{\partial^2 u}{\partial y^2} = X\,Y''$$

etc.

Example 2: Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = 4\,\frac{\partial u}{\partial y}$$

$$\text{If} \quad u(x,y) = X(x)\,Y(y) \quad \Rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = X''\,Y \\ \dfrac{\partial u}{\partial y} = X\,Y' \end{cases}$$

$$\Rightarrow X''\,Y = 4\,X\,Y'$$

$$\Rightarrow \frac{X''}{4\,X} = \frac{Y'}{Y} = -\lambda \quad \text{(constant)}$$

$$f(x) \qquad f(y)$$

Example 2:

- Case 1: If $\lambda = 0$,

$$\Rightarrow \frac{X''}{4\ X} = \frac{Y'}{Y} =$$

$$\Rightarrow \begin{cases} X'' = \\ Y' = \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = \\ Y(y) = \end{cases}$$

$$\Rightarrow u(x, y) = X(x)\ Y(y)$$

$$=$$

$$=$$

Example 2:

- Case 2: If $\lambda = -\alpha^2 < 0$,

$$\Rightarrow \frac{X''}{4\ X} = \frac{Y'}{Y} =$$

$$\Rightarrow \begin{cases} X'' - \qquad X = 0 \\ Y' - \qquad Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = \\ Y(y) = \end{cases}$$

$$\Rightarrow u(x,y) = X(x)\,Y(y)$$

$$=$$

$$=$$

Example 2:

- Case 3: If $\lambda = \alpha^2 > 0$,

$$\Rightarrow \frac{X''}{4\ X} = \frac{Y'}{Y} =$$

$$\Rightarrow \begin{cases} X'' + \qquad X = 0 \\ Y' + \qquad Y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = \\ Y(y) = \end{cases}$$

$$\Rightarrow u(x, y) = X(x)\ Y(y)$$

$$=$$

$$=$$

IF $u_1(x,y), u_2(x,y), \cdots, u_k(x,y)$ are solutions of homogeneous linear PDE

THEN $u(x,y) = c_1 u_1(x,y) + c_2 u_2(x,y) + \cdots + c_k u_k(x,y)$

is also a solution,

where $c_1, c_2, \cdots, c_k$ are constants.

$$u(x,y) = \sum_{k=1}^{\infty} c_k \, u_k(x,y)$$

12.1: Def 12.1.1: Classification of Equations

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0$$

A, B, C, D, E, F are real numbers

is said to be

if $B^2 - 4AC$ 0 (Equation)

if $B^2 - 4AC$ 0 (Equation)

if $B^2 - 4AC$ 0 (Equation)

Example 3: Classifying Linear 2nd-Order PDEs

(a) $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ $\Rightarrow A = 3, B = 0, C = 0$

$\Rightarrow B^2 - 4AC \quad\quad 0$

$\Rightarrow$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ $\Rightarrow A = 1, B = 0, C = -1$

$\Rightarrow B^2 - 4AC \quad\quad 0$

$\Rightarrow$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\Rightarrow A = 1, B = 0, C = 1$

$\Rightarrow B^2 - 4AC \quad\quad 0$

$\Rightarrow$

Engineering Problems (12.2)

- One-Dimensional Heat Equation:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad k > 0$$

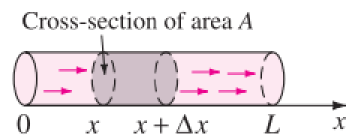
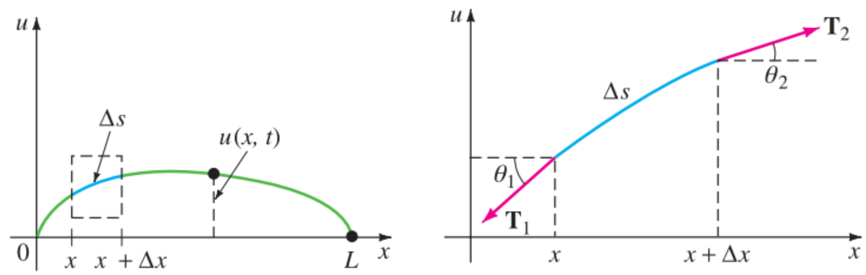


FIGURE 12.2.1 One-dimensional flow of heat

- One-Dimensional Wave Equation:

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$



(a) Segment of string **(b)** Enlargement of segment

FIGURE 12.2.2 Flexible string anchored at $x = 0$ and $x = L$

- Two-Dimensional Form of Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

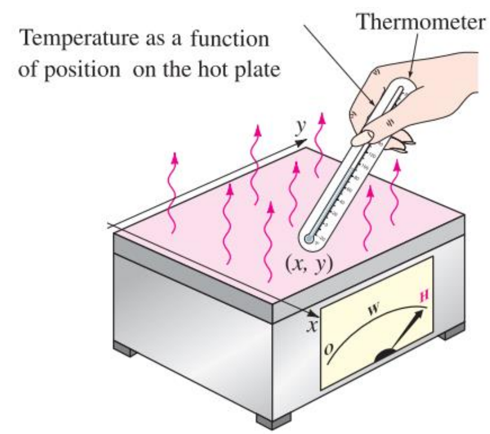


FIGURE 12.2.3 Steady-state temperatures in a rectangular plate

Summary - 12.1: Separable Partial Differential Equations

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u$$

$$u(x,y) = X(x)\, Y(y) \qquad \frac{\partial u}{\partial x} = X' Y \qquad \frac{\partial^2 u}{\partial x^2} = X'' Y$$

$$\frac{\partial u}{\partial y} = X Y' \qquad \frac{\partial^2 u}{\partial y^2} = X Y''$$

- Hyperbolic if $B^2 - 4AC > 0$ (Wave Equation)
- Parabolic if $B^2 - 4AC = 0$ (Heat Equation)
- Elliptic if $B^2 - 4AC < 0$ (Laplace Equation)