

Fall 2019

微分方程 Differential Equations

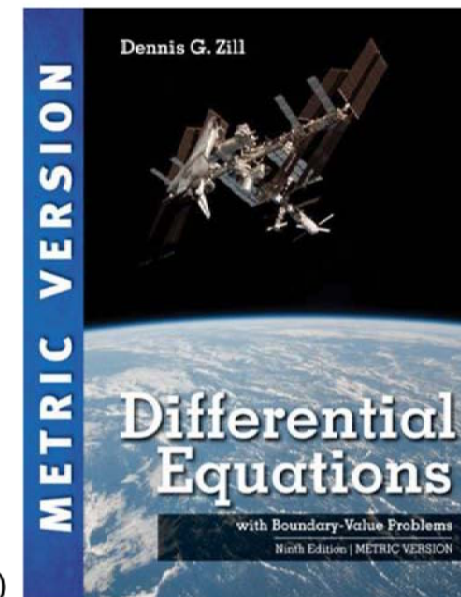
Unit 04.3 Homogeneous Linear Equations with Constant Coefficients

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$$a y''(x) + b y'(x) + c y(x) = 0$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

- 4.1: Linear Differential Equations: Basic Theory
 - 4.1.1: Initial-Value and Boundary-Value Problems
 - 4.1.2: Homogeneous Equations
 - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- ✓ 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- } 4.4: Undetermined Coefficients – Superposition Approach
- } 4.5: Undetermined Coefficients – Annihilator Approach
- ✓ 4.6: Variation of Parameters ✓
- 4.7: Cauchy-Euler Equations
- 4.8: Solving Systems of Linear Equations by Elimination
- 4.9: Nonlinear Differential Equations

 $C_1 \quad C_2 \quad C_3$ $x \quad x^2 \quad x^3$

- Auxiliary Equation:
- Consider a 2nd-order DE:

$$a y''(x) + b y'(x) + c y(x) = 0$$

$$a, b, c \in \mathbb{R}$$

$$a \neq 0$$

- Try a solution

$$y(x) = e^{mx}$$

$$y'(x) = m e^{mx}$$

$$y''(x) = m^2 e^{mx}$$

$$m:$$

$$a(m^2 e^{mx}) + b(m e^{mx}) + c(e^{mx}) = 0$$

$$\underbrace{e^{mx}}_{\neq 0} (a m^2 + b m + c) = 0$$

$$\Rightarrow \underline{a m^2 + b m + c = 0}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Case 1: Two distinct real roots, m_1, m_2 $m_1 \neq m_2$

$\{ \underline{e^{m_1 x}}, \underline{e^{m_2 x}} \}$: linearly independent
fundamental set of solutions

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

4.3: Three Cases

- Case 2: Repeated real roots, $m_1 = m_2 = -\frac{b}{2a}$

$$\{ \underbrace{e^{m_1 x}}_{=}, e^{m_2 x} \} : \text{linearly dependent}$$

$$y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$y_2(x) = y_1 u = y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx = e^{m_1 x} \int \frac{e^{-\int \frac{b}{2a} dx}}{e^{2m_1 x}} dx$$

$$= \underbrace{e^{m_1 x}}_{=}$$

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

$$= c_1 \underbrace{e^{(-\frac{b}{2a})x}}_{x^2} + c_2 \underbrace{x e^{(-\frac{b}{2a})x}}_{x^3}$$

- Case 3: Conjugate complex roots, m_1, m_2

$$m_1 = \alpha + \beta j$$

$$m_2 = \alpha - \beta j$$

Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\beta} + e^{-i\beta} = 2 \cos \beta$$

$$e^{i\beta} - e^{-i\beta} = 2i \sin \beta$$

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{(\alpha + \beta j)x} + c_2 e^{(\alpha - \beta j)x}$$

$$= e^{\alpha x} e^{\beta j x} + e^{\alpha x} e^{-\beta j x}$$

$$= e^{\alpha x} (e^{j\beta x} + e^{-j\beta x})$$

$$= e^{\alpha x} 2 \cos \beta x$$

$$= 2 \cos(\beta x) e^{\alpha x}$$

$$\cos \beta x = \frac{e^{j\beta x} + e^{-j\beta x}}{2}$$

$$\sin \beta x = \frac{e^{j\beta x} - e^{-j\beta x}}{2j}$$

$$2j \sin(\beta x) e^{\alpha x}$$

$$= c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$= e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

4.3: Example 1

$$(a) \quad 2y'' - 5y' - 3y = 0$$

$$y = e^{mx} \quad y' = m e^{mx} \quad y'' = m^2 e^{mx}$$

$$2(m^2 e^{mx}) - 5(m e^{mx}) - 3(e^{mx}) = 0$$

$$e^{mx} (2m^2 - 5m - 3) = 0$$

$$e^{mx} \neq 0, \quad 2m^2 - 5m - 3 = 0 \Rightarrow m_{1,2} = -\frac{1}{2}, 3$$

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

$$(b) \quad y'' - 10y' + 25y = 0$$

$$y = e^{mx}, \quad y' = m e^{mx}, \quad y'' = m^2 e^{mx}$$

$$m^2 e^{mx} - 10m e^{mx} + 25 e^{mx} = 0$$

$$e^{mx} (m^2 - 10m + 25) = 0$$

$$\Rightarrow m_{1,2} = 5, 5$$

$$e^{mx} \neq 0, \quad m^2 - 10m + 25 = 0$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

$$(c) y'' + 4y' + 7y = 0$$

$$y = e^{mx} \quad y' = \quad y'' =$$

$$m^2 e^{mx} + 4m e^{mx} + 7 e^{mx} = 0$$

$$e^{mx} (m^2 + 4m + 7) = 0$$

$$e^{mx} \neq 0 \quad m^2 + 4m + 7 = 0$$

$$\Rightarrow m_{1,2} = -2 \pm \sqrt{3}i \Rightarrow y = e^{-2x} \left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x \right)$$

4.3: Two Special Equations

$$\underline{y''} + \underline{k^2} \underline{y} = 0$$

$$y = e^{mx} \quad y' \quad y''$$

$$m^2 e^{mx} + k^2 e^{mx} = 0$$

$$e^{mx} (m^2 + k^2) = 0$$

$$e^{mx} \neq 0 \Rightarrow m^2 + k^2 = 0$$

$$m = \pm k j$$

$$y = c_1 \underline{\cos kx} + c_2 \underline{\sin kx}$$

$$\underline{y''} - \underline{k^2} \underline{y} = 0$$

⋮

$$m^2 - k^2 = 0$$

$$m = \pm k$$

$$y = c_1 \underline{e^{kx}} + c_2 \underline{e^{-kx}}$$

$$= e^{kx} + e^{-kx}$$

$$= 2 \cosh(kx)$$

$$= 2 \sinh(kx)$$

$$y = c_1 \cosh(kx) + c_2 \sinh(kx)$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$\Rightarrow$ $y = e^{mx}$ $y' = m e^{mx}$ $y'' = m^2 e^{mx}$ $\dots$ $y^{(n-1)} = m^{n-1} e^{mx}$
 $y^{(n)} = m^n e^{mx}$

$e^{mx} \left(\underline{a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0} \right) = 0$

- Case 1: All distinct real roots

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

4.3: Higher-Order Equations

- Case 2: k repeated real roots, $k \leq n$

$$y(x) = c_1 \underbrace{e^{mx}}_{x^0} + c_2 \underbrace{e^{mx}}_{(x)^1} + c_3 \underbrace{e^{mx}}_{(x^2)^1} + \dots + c_k \underbrace{e^{mx}}_{(x^{k-1})^1} + c_{k+1} e^{m_{k+1}x} + \dots + c_n e^{m_n x}$$

m, m, m, m_{k+1}, m_k

- Case 3: Complex roots in conjugate pair

$$y(x) = e^{\alpha x} \cos \beta x$$

$$e^{\alpha x} \sin \beta x$$

$$x e^{\alpha x} \cos \beta x$$

$$x e^{\alpha x} \sin \beta x$$

$$x^2 e^{\alpha x} \cos \beta x$$

$$x^2 e^{\alpha x} \sin \beta x$$

$$(\alpha \pm \beta i)^3$$

$$e^{\alpha x} (\cos \beta x + \sin \beta x)$$

- Consider a 2nd-order DE:

$$a y''(x) + b y'(x) + c y(x) = 0 \quad a, b, c \in \mathbb{R}$$
$$a \neq 0$$

- Try a solution

$$y(x) = e^{mx}$$

$$y'(x) = m e^{mx} \Rightarrow (a m^2 + b m + c) e^{mx} = 0$$

$$y''(x) = m^2 e^{mx}$$

- Case 1: Two distinct real roots, m_1, m_2

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

- Case 2: Repeated real roots, $m_1 = m_2$

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

- Case 3: Conjugate complex roots, $m_{1,2} = a \pm ib$

$$y(x) = e^{ax} (c_1 \cos bx + c_2 \sin bx)$$