Fall 2019

微分方程
Differential Equations

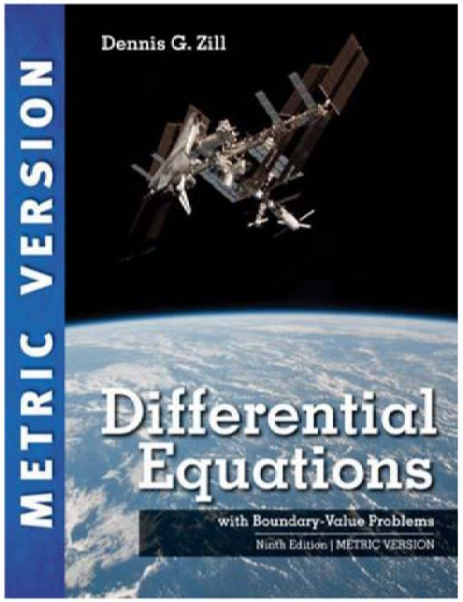
Unit 04.3
Homogeneous Linear Equations with Constant Coefficients

Feng-Li Lian

NTU-EE

Sep19 – Jan20

$$a\, y''(x) + b\, y'(x) + c\, y(x) = 0$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

$c_1 \quad c_2 \quad c_3$

$x \quad x^2 \quad x^3$

- Auxiliary Equation:

- Consider a 2nd-order DE:

$$a\,y''(x) + b\,y'(x) + c\,y(x) = 0$$

$a, b, c \in \mathbb{R}$

$a \neq 0$

- Try a solution $\boxed{y(x) = e^{mx}}$

m :

$y'(x) = m e^{mx}$

$y''(x) = m^2 e^{mx}$

$$a(m^2 e^{mx}) + b(m e^{mx}) + c(e^{mx}) = 0$$

$$\underbrace{e^{mx}}_{\neq 0\ \forall x}(a m^2 + b m + c) = 0$$

$$\Rightarrow \boxed{a m^2 + b m + c = 0}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Case 1: Two distinct real roots, m_1, m_2 $\quad m_1 \neq m_2$

$\{ e^{m_1 x}, e^{m_2 x} \}$: linearly independent
fundamental set of solutions

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

- Case 2: Repeated real roots, $m_1 = m_2$ $= \frac{-b}{2a}$

$\{ e^{m_1 x}, e^{m_2 x} \}$: linearly dependent

$y_1(x) =$ $e^{m_1 x} \doteq e^{m_2 x} = e^{-\frac{b}{2a}x}$

$y_2(x) =$ $y_1 u = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx = e^{m_1 x} \int \frac{e^{-\int \frac{b}{a} dx}}{e^{2 m_1 x}} dx$

$e^{2(-\frac{b}{2a})x}$

$= e^{m_1 x} x$

$y(x) =$ $c_1 y_1 + c_2 y_2 = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

$= c_1 \boxed{e^{(-\frac{b}{2a})x}} + c_2 x \boxed{e^{(-\frac{b}{2a})x}}$

$x^2(\quad)$ x^3

- Case 3: Conjugate complex roots, m_1, m_2

$m_1 \neq m_2$ $m_1 = \alpha + \beta j$

$m_2 = \alpha - \beta j$

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{(\alpha+\beta j)x} + c_2 e^{(\alpha-\beta j)x}$

$= e^{\alpha x} e^{\beta j x} + e^{\alpha x} e^{-\beta j x}$

$= e^{\alpha x} \left(e^{j\beta x} + e^{-j\beta x} \right)$

$= e^{\alpha x}\, 2 \cos \beta x$

$= 2 \boxed{\cos(\beta x)}\, e^{\alpha x}$

$2j \boxed{\sin(\beta x)}\, e^{\alpha x}$

Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$e^{i\beta} + e^{-i\beta} = 2\cos\beta$$

$$e^{i\beta} - e^{-i\beta} = 2i\sin\beta$$

$\cos \beta x = \dfrac{e^{j\beta x} + e^{-j\beta x}}{2}$

$\sin \beta x = \dfrac{e^{j\beta x} - e^{-j\beta x}}{2j}$

$$= c_1 \boxed{e^{\alpha x} \cos \beta x} + c_2 \boxed{e^{\alpha x} \sin \beta x}$$

$$= e^{\alpha x} \left(\underline{c_1 \cos \beta x} + \underline{c_2 \sin \beta x} \right)$$

(a) $2y'' - 5y' - 3y = 0$

$y = e^{mx} \quad y' = me^{mx} \quad y'' = m^2 e^{mx}$

$2(m^2 e^{mx}) - 5(m e^{mx}) - 3(e^{mx}) = 0$

$e^{mx}\left(\underline{2m^2 - 5m - 3}\right) = 0$

$e^{mx} \neq 0, \quad 2m^2 - 5m - 3 = 0 \Rightarrow m_{1,2} = -\frac{1}{2}, 3$

$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$

(b) $y'' - 10y' + 25y = 0$

$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$

$m^2 e^{mx} - 10 m e^{mx} + 25 e^{mx} = 0$

$e^{mx}(m^2 - 10m + 25) = 0$

$e^{mx} \neq 0, \quad m^2 - 10m + 25 = 0 \Rightarrow m_{1,2} = 5, 5$

$y = c_1 \boxed{e^{5x}} + c_2 \boxed{x e^{5x}}$

(c) $y'' + 4y' + 7y = 0$

$y = e^{mx}$ $y' =$ $y'' =$

$m^2 e^{mx} + 4 m e^{mx} + 7 e^{mx} = 0$

$e^{mx} (m^2 + 4m + 7) = 0$

$e^{mx} \neq 0$ $m^2 + 4m + 7 = 0$

$\Rightarrow m_{1,2} = -2 \pm \sqrt{3} j \Rightarrow y = e^{-2x} \left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x \right)$

$y'' + k^2 y = 0$

$y = e^{mx}$ $\quad y'$ $\quad y''$

$m^2 e^{mx} + k^2 e^{mx} = 0$

$e^{mx}(m^2 + k^2) = 0$

$e^{mx} \neq 0 \Rightarrow m^2 + k^2 = 0$

$m = \pm k j$

$$\boxed{y = c_1 \cos kx + c_2 \sin kx}$$

$y'' - k^2 y = 0$

$\vdots$

$m^2 - k^2 = 0$

$m = \pm k$

$y = c_1 e^{kx} + c_2 e^{-kx}$

$c_1 = 1 \; (1)$, $c_2 = 1 \; (-1)$

$= e^{kx} + e^{-kx}$

$= 2\cosh(kx)$

$= 2\sinh(kx)$

$$\boxed{= c_1 \cosh(kx) + c_2 \sinh(kx)}$$

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y = 0$$

$\Rightarrow$ $y = e^{mx}$ $\quad y' = m e^{mx}$ $\quad y^{(2)} = m^2 e^{mx}$ $\quad \cdots \quad y^{(n-1)} = m^{n-1} e^{mx}$

$y^{(n)} = m^n e^{mx}$

$e^{mx}\left(a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0\right) = 0$

- Case 1: All distinct real roots

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

- Case 2: k repeated real roots, $k \leq n$

$m, m, m, m_{k+1}, m_{k+2}$

$$y(x) = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + c_k x^{k-1} e^{mx} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}$$

- Case 3: Complex roots in conjugate pair

$(\alpha \pm \beta i)^3$

$e^{\alpha x}(\cos\beta x + \sin\beta x)$

$$y(x) = e^{\alpha x}\cos\beta x \quad e^{\alpha x}\sin\beta x \quad x e^{\alpha x}\cos\beta x \quad x e^{\alpha x}\sin\beta x \quad x^2 e^{\alpha x}\cos\beta x \quad x^2 e^{\alpha x}\sin\beta x$$

- Consider a 2nd-order DE:

$$a\, y''(x) + b\, y'(x) + c\, y(x) = 0$$

$a, b, c \in \mathbb{R}$

$a \neq 0$

- Try a solution

$$y(x) = e^{mx}$$
$$y'(x) = m\, e^{mx}$$
$$y''(x) = m^2\, e^{mx}$$

$$\Rightarrow \left(a\, m^2 + b\, m + c\right) e^{mx} = 0$$

- Case 1: Two distinct real roots, m_1, m_2

$$y(x) = c_1\, e^{m_1 x} + c_2\, e^{m_2 x}$$

- Case 2: Repeated real roots, $m_1 = m_2$

$$y(x) = c_1\, e^{m_1 x} + c_2\, x\, e^{m_1 x}$$

- Case 3: Conjugate complex roots, $m_{1,2} = a \pm ib$

$$y(x) = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$$