Fall 2019

微分方程
Differential Equations

Unit 04.5
Undetermined Coefficients – Annihilator

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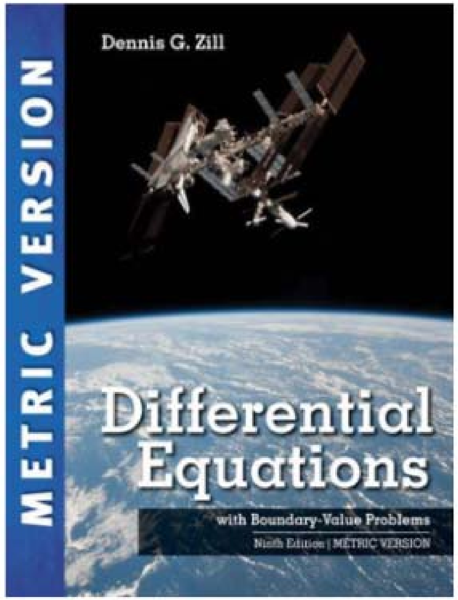

Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

$$\frac{d}{dx}\left(y'' + 3y' + 2y\right) = \frac{d}{dx}\left(4x^2\right) =$$

$$\frac{d}{dx} = D$$

$$\left(\frac{d}{dx}\left(y'' + 3y' + 2y\right)\right) = \left(4\times 2\, x = 8x\right)$$

$$\frac{d}{dx^2}\left(y'' + 3y' + 2y\right) = 8$$

$$\frac{d}{dx^3}\left(y'' + 3y' + 2y\right) = 0$$

$$D^3\left(y'' + 3y' + 2y\right) = D^3\left(4x^2\right) = 0$$

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} = 0$$

y_c

$$e^{m_1 x} + e^{m_2 x}$$

$$e^{m_3 x} \quad e^{m_4 x} \quad e^{m_5 x}$$

y_p

$$2\left(y'' + 3y' + 2y\right) = 2e^{2x}\quad \times (2) = 0$$

$$D\left(y'' + 3y' + 2y\right) = 2e^{2x}$$

$$(D-2)\left(y'' + 3y' + 2y\right) = (D-2)\,e^{2x}$$

$$= De^{2x} - 2e^{2x}$$

$$= 2e^{2x} - 2e^{2x} = 0$$

⇒ purpose:

- Annihilator Operator: $D = \frac{d}{dx}$

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

  - L : a linear differential operator with constant coefficients
  - $f(x)$: a sufficiently differentiable function

- IF $L\big(f(x)\big) = a_n D^n\big(f(x)\big) + a_{n-1} D^{n-1}\big(f(x)\big) \cdots = 0$

- THEN L is said to be an annihilator of function $f(x)$

4.5: Examples

(1) 1 $\quad x$ $\quad x^2$ $\quad \cdots$ $\quad x^{n-1}$

D $\quad D^2$ $\quad D^3$ $\quad D^n$

(2) e^{ax} $\quad xe^{ax}$ $\quad x^2e^{ax}$ $\quad \cdots$ $\quad x^{n-1}e^{ax}$

$(D-a)$ $\quad (D-a)^2$ $\quad (D-a)^3$ $\quad (D-a)^n$

(3) $e^{ax}\cos bx$ $\quad xe^{ax}\cos bx$ $\quad \cdots$ $\quad x^{n-1}e^{ax}\cos bx$

$e^{ax}\sin bx$ $\quad xe^{ax}\sin bx$ $\quad \cdots$ $\quad x^{n-1}e^{ax}\sin bx$

$\boxed{(D^2+\alpha D+\beta)}$ $\quad (D^2+\alpha D+\beta)^2$ $\quad (D^2+\alpha D+\beta)^n$

$D = +a \pm jb$

$L(f(x)) = 0$

4.5: Example 3

$$D^3(\boxed{y'' + 3y' + 2y}) = D^3(4x^2) = 0$$

y_c

$$\boxed{m^2 + 3m + 2 = 0}$$

$$(m+2)(m+1) = 0$$

$$m = -2, -1$$

$L = D^3$

$P(D) =$

$$\begin{cases} L(f(x)) = g(x) \\ L(f(x)) = \end{cases}$$

$\Rightarrow$

- $y_c = \boxed{c_1 e^{-2x} + c_2 e^{-x}}$

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} = 0$$

$$\Rightarrow m^5 + 3m^4 + 2m^3 = 0$$

$$m^3(\boxed{m^2 + 3m + 2}) = 0$$

$$\Rightarrow m_{1,2} = -1, -2, \quad m_{3,4,5} = 0, 0, 0$$

$e^{mx} \Rightarrow e^{0x}, \quad e^{0x}, \quad e^{0x}$

$m_{3,4,5}$

$1 \qquad x \cdot 1 \qquad x^2 \cdot 1$

$y_p = C_3 \cdot 1 + C_4 x + C_5 x^2$

4.5: Example 5

$(D^2+1)(y'' + y) = x\cos x \quad (D^2+1)^2$

$(D^2+1)\ \cos x$

$(D^2+1)^2\ \cancel{x}\cos x$

- y_c

$(D^2+1)^2(y''+y) = 0$

$(D^2+1)y$

- y_p

$(D^2+1)^3 y = 0$

$\Rightarrow (m^2+1)^3 = 0 \Rightarrow \boxed{m = \pm i, \pm i, \pm i}$

$y(x) = c_1\cos x + c_2\sin x$

$c_3 x\cos x + c_4 x\sin x$

$c_5 x^2\cos x + c_6 x^2\sin x$

$$y'' - 2y' + y = 10e^{-2x}\cos(x)$$

$$(D^2+4D+5)e^{-2x}\cos(x) \leftrightarrow -2 \pm j$$

$$\downarrow$$

$$D^2+4D+5$$

$$= 0$$

$$(D^2+4D+5)(D^2-2D+1)y = 0$$

$$y = e^{mx} \Rightarrow (m^2+4m+5)(m^2-2m+1) = 0$$

$$m = -2 \pm j,\ 1,\ 1$$

$$y = \underbrace{c_1 e^x + c_2 x e^x}_{y_c} + \underbrace{c_3 e^{-2x}\cos x + c_4 e^{-2x}\sin x}_{y_p}$$

$$y'' + 4\,y' - 2\,y = 2x^2 + 6 + 10\sin 3x - 3e^{-2x}$$

$$\Rightarrow y = y_c + y_p$$

4.4: $\Rightarrow y_p = (Ax^2+Bx+C) + (E\cos 3x + F\sin 3x) + (Ge^{-2x})$

4.5: $\Rightarrow P(D)\left(y'' + 4\,y' - 2\,y\right)$

$$= P(D)\left(2x^2 + 6 + 10\sin 3x - 3e^{-2x}\right) = 0$$

$e.g., \Rightarrow y^{(4)} - 9\,y^{(3)} + 7\,y'' - 6\,y' + 12\,y = 0$

$$\Rightarrow y = y_c + y_p$$