

Fall 2019

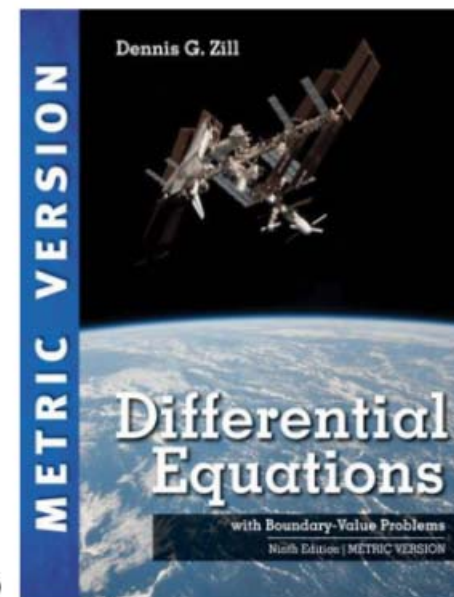
# 微分方程 Differential Equations

## Unit 04.5 Undetermined Coefficients – Annihilator

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Figures and images used in these lecture notes are adopted from  
[Differential Equations with Boundary-Value Problems](#), 9th Ed., D.G. Zill, 2018 (Metric Version)

- 4.1: Linear Differential Equations: Basic Theory
  - 4.1.1: Initial-Value and Boundary-Value Problems
  - 4.1.2: Homogeneous Equations
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- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach
- **4.5: Undetermined Coefficients – Annihilator Approach**
- 4.6: Variation of Parameters
- 4.7: Cauchy-Euler Equations
- 4.8: Solving Systems of Linear Equations by Elimination
- 4.9: Nonlinear Differential Equations

## 4.5: Undetermined Coefficients – Annihilator Approach

$$\frac{d}{dx} (y'' + 3y' + 2y) = \frac{d}{dx} (4x^2) =$$

$$\left( \frac{d}{dx} (y'' + 3y' + 2y) \right) = (4x^2 \cdot 2x = \underline{8x})$$

$$\frac{d}{dx^2} (y'' + 3y' + 2y) = 8$$

$$\frac{d}{dx^3} (y'' + 3y' + 2y) = 0$$

$$D^3 (y'' + 3y' + 2y) = D^3 (4x^2) = 0$$

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} = 0$$

$$\frac{d}{dx} = D$$

$y_c$

$$e^{m_1 x} + e^{m_2 x}$$

$$e^{m_3 x} \quad e^{m_4 x} \quad e^{m_5 x}$$

$y_p$

$$\begin{aligned} 2(y'' + 3y' + 2y) &= 2e^{2x} X'(2) = 0 \\ D(y'' + 3y' + 2y) &= 2e^{2x} \end{aligned}$$

$$\begin{aligned} (D-2)(y'' + 3y' + 2y) &= (D-2)e^{2x} \\ &= D e^{2x} - 2e^{2x} \\ &= 2e^{2x} - 2e^{2x} = 0 \end{aligned}$$

⇒ purpose:

- Annihilator Operator:  $D = \frac{d}{dx}$

$$\underline{L} = \underline{a_n} D^n + \underline{a_{n-1}} D^{n-1} + \dots + \underline{a_1} D + \underline{a_0}$$

- $\underline{L}$ : a linear differential operator with constant coefficients

- $\underline{f(x)}$ : a sufficiently differentiable function

- IF  $\underline{L(f(x))} = \underline{a_n D^n(f(x))} + \underline{a_{n-1} D^{n-1}(f(x))} + \dots = 0$

- THEN  $\underline{L}$  is said to be an annihilator of function  $f(x)$

$$(1) \quad \frac{1}{D} \quad x \quad x^2 \quad \dots \quad x^{n-1}$$

$$D \quad D^2 \quad D^3 \quad \dots \quad D^n$$

$$(2) \quad e^{ax} \quad x e^{ax} \quad x^2 e^{ax} \quad \dots \quad x^{n-1} e^{ax}$$

$$(D-a) \quad (D-a)^2 \quad (D-a)^3 \quad \dots \quad (D-a)^n$$

$$(3) \quad e^{ax} \cos bx \quad x e^{ax} \cos bx \quad \dots \quad x^{n-1} e^{ax} \cos bx$$

$$e^{ax} \sin bx \quad x e^{ax} \sin bx \quad \dots \quad x^{n-1} e^{ax} \sin bx$$

$$(D^2 + \alpha D + \beta) \quad (D^2 + \alpha D + \beta)^2 \quad (D^2 + \alpha D + \beta)^n$$

$$D = \pm a \pm jb$$

$$L(f(x)) = 0$$

$$D^3(y'' + 3y' + 2y) = D^3(4x^2) = 0$$

$$L = D^3$$

$$P(D) =$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2, -1$$

$$L(f(x)) = g(x)$$

$$L(f(x)) =$$

 $\Rightarrow$ 

$$y_c = c_1 e^{-2x} + c_2 e^{-x}$$

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} = 0$$

$$\Rightarrow m^5 + 3m^4 + 2m^3 = 0$$

$$m^3(m^2 + 3m + 2) = 0$$

$$\Rightarrow m_{1,2} = -1, -2, \quad m_{3,4,5} = 0, 0, 0$$

$$e^{mx} \Rightarrow \begin{matrix} e^{0x} \\ | \\ 1 \end{matrix}, \begin{matrix} e^{0x} \\ | \\ x \cdot 1 \end{matrix}, \begin{matrix} e^{0x} \\ | \\ x^2 \cdot 1 \end{matrix}$$

$m_{3,4,5}$

$$y_p = \underline{c_3 \cdot 1 + c_4 x + c_5 x^2}$$



# 4.5: Example 5

$$(D^2+1)(y''+y) = (x \cos x) (D^2+1)^2 (D^2+1) \cos x$$

$(D^2+1)^2 \cos x$

•  $y_c$

$$(D^2+1)^2 (y''+y) = 0$$

$$(D^2+1)y$$

•  $y_p$

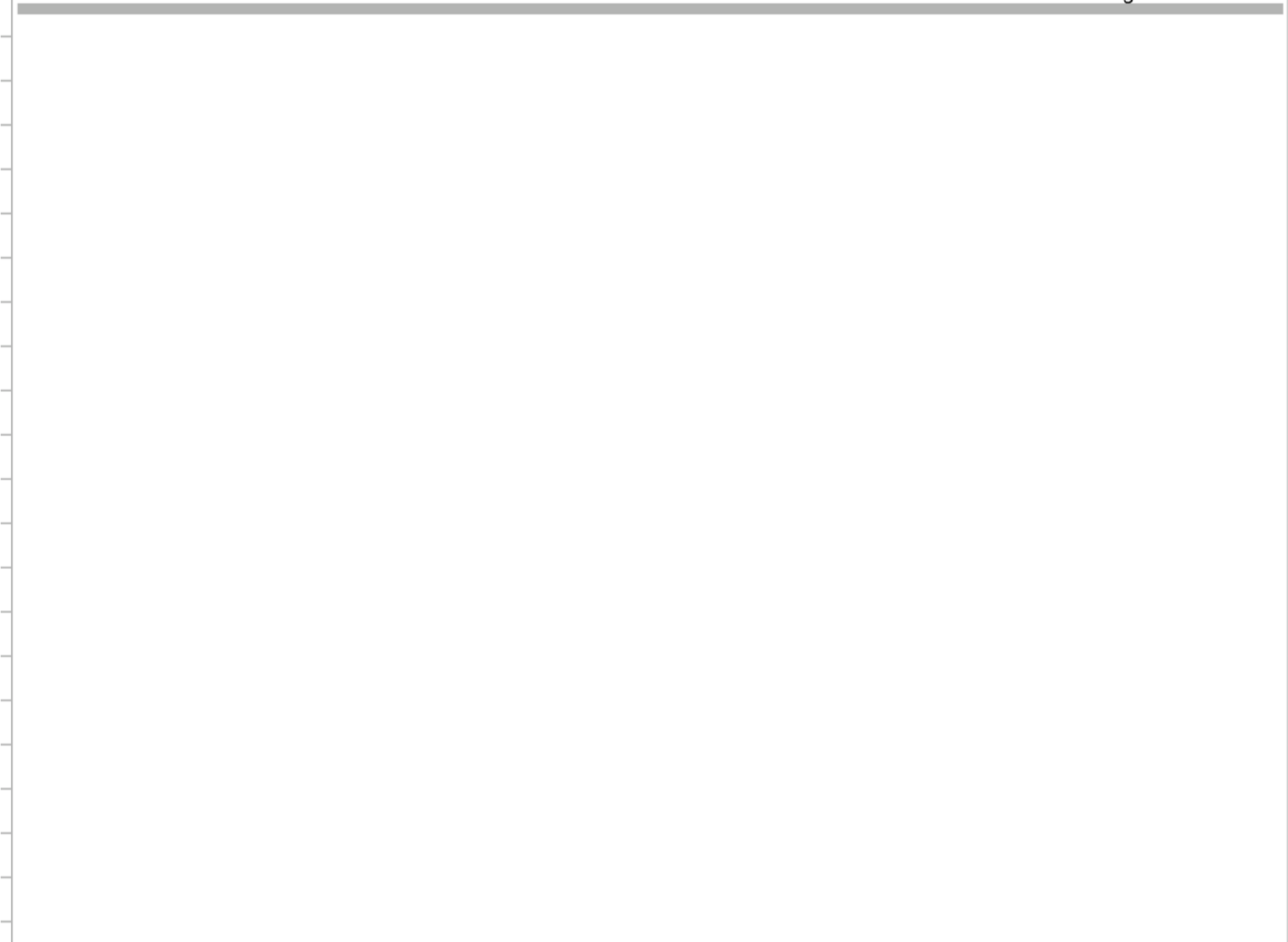
$$(D^2+1)^3 y = 0$$

$$\Rightarrow (m^2+1)^3 = 0 \Rightarrow m = \pm i, \pm i, \pm i$$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$C_3 x \cos x + C_4 x \sin x$$

$$C_5 x^2 \cos x + C_6 x^2 \sin x$$



## 4.5: Example 6

$$y'' - 2y' + y = 10e^{-2x} \cos(x)$$

$$(D^2 + 4D + 5)e^{-2x} \cos(x) \leftrightarrow -2 \pm j$$

$$D^2 + 4D + 5$$

$$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0$$

$$y = e^{mx} \Rightarrow (m^2 + 4m + 5)(m^2 - 2m + 1) = 0$$

$$m = -2 \pm j, 1, 1$$

$$y = \underbrace{C_1 e^x + C_2 x e^x}_{y_c} + \underbrace{C_3 e^{-2x} \cos x + C_4 e^{-2x} \sin x}_{y_p}$$

$$y'' + 4y' - 2y = 2x^2 + 6 + 10\sin 3x - 3e^{-2x}$$

$$\Rightarrow y = y_c + y_p$$

$$\text{4.4: } \Rightarrow y_p = (Ax^2 + Bx + C) + (E \cos 3x + F \sin 3x) + (Ge^{-2x})$$

$$\begin{aligned} \text{4.5: } \Rightarrow & \underbrace{P(D)}_{\text{annihilator}} \underbrace{(y'' + 4y' - 2y)}_{\text{LHS}} \\ & = \underbrace{P(D)}_{\text{annihilator}} \underbrace{(2x^2 + 6 + 10\sin 3x - 3e^{-2x})}_{\text{RHS}} \quad (= 0) \end{aligned}$$

$$\text{e.g., } \Rightarrow \underline{y^{(4)} - 9y^{(3)} + 7y'' - 6y' + 12y = 0}$$

$$\Rightarrow y = y_c + y_p$$