Fall 2019

微分方程
Differential Equations

Unit 04.7
Cauchy-Euler Equations

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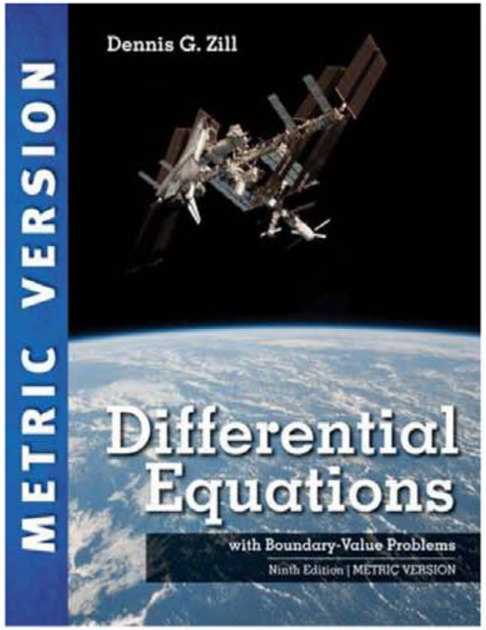
$$ax^2y'' + bxy' + cy = 0$$

$$y(x) = x^m$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

Outline

4.7: Cauchy-Euler Equations

$2x^9$ $\quad x^{10}$ $\quad n+1$

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x)$$

x^{10}

$$a_i \in \mathbb{R}, \quad I = (0, \infty)$$

$y = x^{10}$

$y = e^{3x}$
e^{-5x}

known as a Cauchy-Euler Equation (Equidimensional Eqn)

Augustin-Louis Cauchy (1789-1857)

Leonhard Euler (1707-1783)

- Consider a 2nd-order DE:

$$a x^2 y'' + b x y' + c y = 0$$

$x^2 \neq 0$

$I = (0, \infty)$
$(0, -\infty)$

- Assume $y(x) = x^m \quad \times c$

$y'(x) = m x^{m-1} \quad \times bx \quad = 0$

$y''(x) = m(m-1) x^{m-2} \quad \times a x^2$

$a m(m-1) x^m + bm x^m + c x^m = 0$

$$x^m \left[a m(m-1) + bm + c \right] = 0$$

$$a m(m-1) + bm + c = 0$$ Auxiliary Eqn

$$a m^2 + (b-a) m + c = 0$$

$$m = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a} \Rightarrow \frac{-(b-a)}{2a} = \frac{a-b}{2a}$$

- Case I: distinct real roots: m_1, m_2

$$y_c(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

- Case II: repeated real roots: $m_1 = m_2$

$m = \frac{a-b}{2a}, \frac{a-b}{2a} = k$

$y = e^{mx}$

$$c\begin{cases} y_1(x) = x^k = y_1 \\ y_2(x) = u(x)\,x^k = u y_1 \end{cases}$$

$x e^{kx}$

$bx\; y_2'(x) = u' x^k + u k x^{k-1} = u' y_1 + u y_1'$

$ax^2\; y_2''(x) = u'' x^k + u'(k) x^{k-1} + u' k x^{k-1} + u (k)(k-1) x^{k-2}$

$= u'' y_1 + 2u' y_1' + u y_1''$

$$ax^2\left(u''y_1 + 2u'y_1' + uy_1''\right) + bx\left(u'y_1 + uy_1'\right) + cx\left(uy_1\right) = 0$$

$$u\left(ax^2y_1'' + bxy_1' + cy_1\right) \Rightarrow 0$$
$$+\,u'\left(2ax^2y_1' + bxy\right)$$
$$+\,u''\left(ax^2y_1\right) = 0$$

$$u'\left(2ax^2mx^{m-1} + bx\,x^m\right) + u''\left(ax^2x^m\right) = 0$$

$$u'\left(2amx^{m+1} + bx^{m+1}\right) + u''\left(ax^{m+2}\right) = 0$$

$$u'\left(2am + b\right) + u''ax = 0$$

$$u'' + \frac{1}{x}\left(\boxed{\frac{2am+b}{a}}\right)u' = 0$$

$$\frac{2am+b}{a} = \frac{2a\frac{a-b}{2a}+b}{a} = \frac{a}{a} = 1$$

$$u'' + \frac{1}{x} u' = 0 \Rightarrow u' = \frac{1}{x}$$

$$w = u' \qquad u = \ln x$$

$$y_2 = u y_1 = (\ln x) x^k \qquad m = k, k$$

$$y = c_1 y_1 + c_2 y_2 = c_1 x^k + c_2 (\ln x) x^k$$

$$\left(\boxed{y = c_1 e^{kx} + c_2 x e^{kx}} \right)$$

- Case III: conjugate complex roots

$$\begin{cases} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{cases} \qquad \alpha, \beta \in \mathbb{R}, \quad \beta > 0$$

$$\Rightarrow \quad x^{m_1} = x^{(\alpha+i\beta)} = x^{\alpha} x^{i\beta} = x^{\alpha}\left(e^{\ln x}\right)^{i\beta}$$

$$= x^{\alpha} e^{(i\beta)\ln x} = x^{\alpha} e^{i(\beta \ln x)}$$

$$= x^{\alpha}\left[\cos(\beta \ln x) + i \sin(\beta \ln x)\right]$$

$$\Rightarrow \quad x^{m_2} = x^{\alpha}\left[\cos(\beta \ln x) - i \sin(\beta \ln x)\right]$$

$$\Rightarrow \quad y(x) = c_1 x^{m_1} + c_2 x^{m_2} = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$= C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$$

$$\boxed{a\,x^2\,y'' + b\,x\,y' + c\,y = 0}$$

$y(x) \quad x$

$y(t) \quad t$

$y = e^{mx}$

- Let $x = e^t$ $\quad t = \ln x$

$y(x) \longrightarrow y(t)$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right) = e^{-t} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$a\cancel{x^2}\left(-\frac{1}{\cancel{x^2}}\frac{dy}{dt} + \frac{1}{\cancel{x^2}}\frac{d^2y}{dt^2}\right) + b\cancel{x}\left(\frac{1}{\cancel{x}}\left(\frac{dy}{dt}\right)\right) + cy = 0$$

$$a\left(\frac{d^2y}{dt^2}\right) + (b-a)\left(\frac{dy}{dt}\right) + cy = 0$$

$$\boxed{a\,y'' + (b-a)\,y' + c\,y = 0} \qquad y(t) = e^{mt}$$

- auxiliary eqn: $a m^2 + (b-a) m + c = 0$

- Case I: m_1, m_2 distinct real roots

$\boxed{x = e^t}$

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

- Case II: $m_1 = m_2$

$$y(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

$$y(x) = c_1 x^{m_1} + c_2 (\ln x) x^{m_1}$$

- Case III: $m_{1,2} = \alpha \pm i\beta$

$$y(t) = c_1 \underline{e}^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

$$y(x) = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$$

$$\cancel{x}^{k} \qquad e^{k\cancel{x}}$$

$$x^2\, y'' - 3\, x\, y' + 3\, y = 2\, x^4\, e^x$$

- auxiliary eqn. $y(x) = x^m$

$$x^2\, m(m-1)\, x^{m-2} - 3\, x\, m x^{m-1} + 3 x^m = 0$$

$$x^m \left[m(m-1) - 3m + 3 \right] = 0$$

$$m^2 - m - 3m + 3 \qquad (m-3)(m-1) = 0$$

$$-4m \qquad m = 1, 3$$

$\Rightarrow y_c =$ $c_1 x^1 + c_2 x^3$

$\Rightarrow y_p =$ $u_1 y_1 + u_2 y_2$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = 2x^4 e^x \end{cases}$$

$$\begin{cases} u_1 = -x^2 e^x + 2x e^x - 2 e^x \\ u_2 = e^x \end{cases}$$

$y_p =$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y = y_c + y_p$$

$$= c_1 x + c_2 x^3$$

$$+ (\cancel{-x^2 e^x} + 2xe^x - 2e^x)\cancel{x}$$

$$+ \cancel{e^x x^3}$$

$$= \boxed{c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x}$$

4.7: Another Form of Cauchy-Euler Equation

$$a\,x^2\,y'' + b\,x\,y' + c\,y = 0 \longrightarrow y = x^m$$

$$a\,(x - x_0)^2\,y'' + b\,(x - x_0)\,y' + c\,y = 0$$

$$y = (x - x_0)^m = t^m \qquad x - x_0 = t$$

$$a\,t^2\,\frac{d^2y}{dt^2} + b\,t\,\frac{dy}{dt} + c\,y = 0$$

$$y(t) = c_1 t^{m_1} + c_2 t^{m_2}$$

$$y(x) = c_1\,(x - x_0)^{m_1} + c_2\,(x - x_0)^{m_2}$$

Summary - 4.7: Cauchy-Euler Equations

- Consider a 2nd-order DE: $a\,x^2\,y'' + b\,x\,y' + c\,y = 0$

- Assume

$$y(x) = x^m$$
$$y'(x) = m\,x^{m-1}$$
$$y''(x) = m\,(m-1)\,x^{m-2}$$

$$\Rightarrow a\,m(m-1) + b\,m + c = 0$$

$$\Rightarrow a\,m^2 + (b-a)\,m + c = 0$$

- Case I: distinct real roots: m_1, m_2

$$y(x) = c_1\,x^{m_1} + c_2\,x^{m_2}$$

- Case II: repeated real roots: $m_1 = m_2$

$$y(x) = c_1\,x^{m_1} + c_2\,\ln x\,x^{m_1}$$

- Case III: conjugate complex roots $m_{1,2} = \alpha \pm i\beta$

$$y(x) = x^{\alpha}\Big(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\Big)$$