

Fall 2019

微分方程 Differential Equations

Unit 04.7 Cauchy-Euler Equations

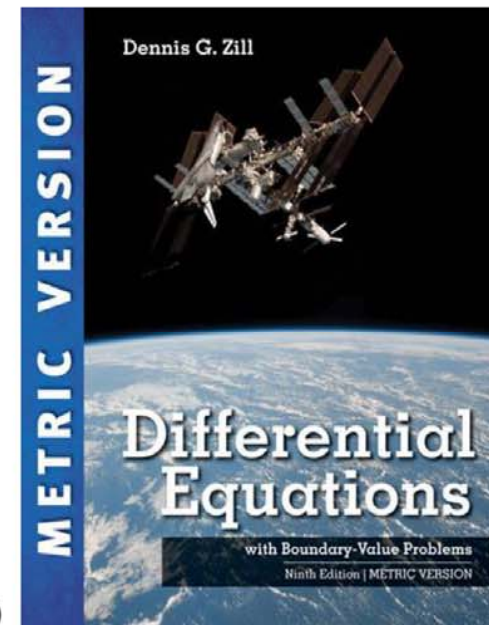
Feng-Li Lian

NTU-EE

Sep19 – Jan20

$$ax^2y'' + bxy' + cy = 0$$

$$y(x) = x^m$$



Figures and images used in these lecture notes are adopted from
Differential Equations with Boundary-Value Problems, 9th Ed., D.G. Zill, 2018 (Metric Version)

- 4.1: Linear Differential Equations: Basic Theory
 - 4.1.1: Initial-Value and Boundary-Value Problems
 - 4.1.2: Homogeneous Equations
 - 4.1.3: Nonhomogeneous Equations
- 4.2: Reduction of Order
- 4.3: Homogeneous Linear Eqns w/ Constant Coefficients
- 4.4: Undetermined Coefficients – Superposition Approach
- 4.5: Undetermined Coefficients – Annihilator Approach
- 4.6: Variation of Parameters
- 4.7: Cauchy-Euler Equations
- 4.8: Green's Functions
- 4.9: Solving Systems of Linear Equations by Elimination
- 4.10: Nonlinear Differential Equations

4.7: Cauchy-Euler Equations

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$$

$a_i \in \mathbb{R}, \quad I = (0, \infty)$

$n+1$
 $y = x^{10}$

$y = e^{\frac{1}{5}x}$

known as a Cauchy-Euler Equation (Equidimensional Eqn)

Augustin-Louis Cauchy (1789-1857)

Leonhard Euler (1707-1783)

- Consider a 2nd-order DE:

$$a x^2 y'' + b x y' + c y = 0$$

$$x^2 \neq 0$$

$$I = \frac{(0, \infty)}{(0, -\infty)}$$

- Assume

$$y(x) = x^m \times C$$

$$y'(x) = m x^{m-1} \times b x = 0$$

$$y''(x) = m(m-1) x^{m-2} \times a x^2$$

$$a m(m-1) x^m + b m x^m + c x^m = 0$$

$$\underline{x^m} [a m(m-1) + b m + c] = 0$$

$$a m(m-1) + b m + c = 0$$

Auxiliary Eqn

$$a m^2 + (b-a)m + c = 0$$

$$m = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a} \Rightarrow \frac{-(b-a)}{2a} = \frac{a-b}{2a}$$

- Case I: distinct real roots: m_1, m_2

$$y_c(x) = \underline{c_1 x^{m_1}} + \underline{c_2 x^{m_2}}$$

- Case II: repeated real roots: $m_1 = m_2$

$$m = \frac{a-b}{2a}, \quad \left(\frac{a-b}{2a} \right) = k$$

$$y = e^{mx}$$

$$\begin{cases} y_1(x) = \underline{x^k} \\ y_2(x) = \underline{u(x) x^k} \end{cases}$$

$$\begin{aligned} &= y_1 \\ &= \underline{u y_1} \end{aligned}$$

$$\underline{x e^{kx}}$$

$$b x y_2'(x) = u' x^k + u k x^{k-1} = \underline{u' y_1 + u y_1'}$$

$$\begin{aligned} a x^2 y_2''(x) &= u'' x^k + u'(k) x^{k-1} + u' k x^{k-1} + u(k)(k-1) x^{k-2} \\ &= \underline{u'' y_1 + 2u' y_1' + u y_1''} \end{aligned}$$

$$ax^2 (u' y_1 + 2u' y_1' + u y_1'') + bx (u' y_1 + u y_1') + cx (u y_1) = 0$$

$$u (ax^2 y_1'' + bx y_1' + c y_1) + u' (2ax^2 y_1' + bx y_1) + u'' (ax^2 y_1) = 0$$

$$u' (2ax^2 m x^{m-1} + bx x^m) + u'' (ax^2 x^m) = 0$$

$$u' (2am x^{m+1} + b x^{m+1}) + u'' (a x^{m+2}) = 0$$

$$u' (2am + b) + u'' a x = 0$$

$$u'' + \frac{1}{ax} (2am + b) u' = 0$$

$$\frac{2am+b}{a} = \frac{2a \frac{ab}{2a} + b}{a} = \frac{a}{a} = 1$$

$$\frac{u''}{w=u'} + \frac{1}{x} u' = 0 \Rightarrow u' = \frac{1}{x}$$

$$u = \ln x$$

$$y_2 = u y_1 = (\ln x) x^k \quad \underline{m=k, k}$$

$$y = c_1 y_1 + c_2 y_2 = c_1 x^k + c_2 (\ln x) x^k$$

$$(y = c_1 e^{kx} + c_2 x e^{kx})$$

- Case III: conjugate complex roots

$$\begin{cases} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{cases} \quad \begin{matrix} \alpha, \beta \in \mathbb{R} \\ \beta > 0 \end{matrix}$$

$$\begin{aligned} \Rightarrow \underline{x^{m_1}} &= x^{(\alpha + i\beta)} = x^\alpha x^{i\beta} = x^\alpha (e^{\ln x})^{i\beta} \\ &= x^\alpha e^{(i\beta)\ln x} = x^\alpha e^{i(\beta \ln x)} \\ &= \underline{x^\alpha} [\cos(\underline{\beta \ln x}) + i \sin(\underline{\beta \ln x})] \end{aligned}$$

$$\Rightarrow \underline{x^{m_2}} = \underline{x^\alpha} [\cos(\underline{\beta \ln x}) - i \sin(\underline{\beta \ln x})]$$

$$\begin{aligned} \Rightarrow y(x) &= C_1 x^{m_1} + C_2 x^{m_2} = x^\alpha [C_1 \cos(\underline{\beta \ln x}) + C_2 \sin(\underline{\beta \ln x})] \\ &= C_1 x^\alpha \cos(\underline{\beta \ln x}) + C_2 x^\alpha \sin(\underline{\beta \ln x}) \end{aligned}$$

4.7: Another Point of View

DE04.7-Cauchy-Euler - 8
Feng-Li Lian © 2019

$$\Rightarrow a x^2 y'' + b x y' + c y = 0$$

$$\begin{matrix} y(x) & \cdot & x \\ y(t) & t \end{matrix}$$

$$y = e^{mx}$$

Let $x = e^t$ $t = \ln x$

$$y(x) \rightarrow y(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{1}{x} = \frac{dy}{dt} e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2y}{dt^2} \frac{dt}{dx}$$

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2y}{dt^2} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2y}{dt^2}$$

$$a x^2 \left(-\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2y}{dt^2} \right) + b x \left(\frac{1}{x} \frac{dy}{dt} \right) + c y = 0$$

$$a \left(\frac{d^2y}{dt^2} \right) + (b-a) \left(\frac{dy}{dt} \right) + c y = 0$$

$$a y'' + (b-a) y' + c y = 0 \quad y(t) = e^{mt}$$

- auxiliary eqn: $a m^2 + (b-a)m + c = 0$

- Case I: m_1, m_2 distinct real roots

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

$$x = e^t$$

- Case II: $m_1 = m_2$

$$y(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

$$y(x) = c_1 x^{m_1} + c_2 (\ln x) x^{m_1}$$

- Case III: $m_{1,2} = \alpha \pm i\beta$

$$y(t) = c_1 \underline{e^{\alpha t}} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

$$y(x) = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$$

~~x^k~~

~~e^{kx}~~

4.7: Example 5

$$\underline{x^2} \underline{y''} - 3 \underline{x} \underline{y'} + 3 \underline{y} = 2 x^4 e^x$$

• auxiliary eqn. $y(x) = x^m$

$$\underline{x^2} m(m-1) \underline{x^{m-2}} - 3 \underline{x} m \underline{x^{m-1}} + 3 \underline{x^m} = 0$$

$$\underline{x^m} \left[\frac{m(m-1) - 3m + 3}{m^2 - m - 3m + 3} \right] = 0$$

$$\underline{m^2 - m - 3m + 3} \quad (m-3)(m-1) = 0$$

$$m = 1, 3$$

$$\Rightarrow y_c = c_1 x^1 + c_2 x^3$$

$$\Rightarrow y_p = \underline{u_1 y_1 + u_2 y_2}$$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = 2x^4 e^x \end{cases}$$

$$\begin{cases} u_1 = -x^2 e^x + 2x e^x - 2e^x \\ u_2 = e^x \end{cases}$$

$y_p =$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y = y_c + y_p$$

$$= c_1 x + c_2 x^3$$

$$+ \underbrace{(-x^2 e^x + 2x e^x - 2e^x)}_{\text{circled X}} \\ + \cancel{e^x x^3}$$

$$= \boxed{c_1 x + c_2 x^3 + x^2 e^x - 2x e^x}$$

4.7: Another Form of Cauchy-Euler Equation

$$a x^2 y'' + b x y' + c y = 0 \longrightarrow y = x^m$$

$$a (\underline{x - x_0})^2 y'' + b (\underline{x - x_0}) y' + c y = 0$$

$$y = (x - x_0)^m = t^m \quad x - x_0 = t$$

$$a (t^2) \frac{d^2 y}{dt^2} + b (t) \frac{dy}{dt} + c y = 0$$

$$y(t) = c_1 t^{m_1} + c_2 t^{m_2}$$

$$y(x) = c_1 (x - x_0)^{m_1} + c_2 (x - x_0)^{m_2}$$

- Consider a 2nd-order DE: $a x^2 y'' + b x y' + c y = 0$

- Assume $y(x) = x^m$

$$y'(x) = m x^{m-1}$$

$$y''(x) = m(m-1) x^{m-2}$$

$$\Rightarrow a m(m-1) + b m + c = 0$$

$$\Rightarrow a m^2 + (b-a) m + c = 0$$

- Case I: distinct real roots: m_1, m_2

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2}$$

- Case II: repeated real roots: $m_1 = m_2$

$$y(x) = c_1 x^{m_1} + c_2 \ln x x^{m_1}$$

- Case III: conjugate complex roots $m_{1,2} = \alpha \pm i\beta$

$$y(x) = x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$$