(4.4)

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$$y'' + 3y = -48x^2e^{3x}$$

$y = e^{mx}$ for AHE

$$(m^2e^{mx}) + 3(e^{mx}) = 0$$

$$e^{mx}(m^2+3) = 0$$

$\because e^{mx} \neq 0\ \forall x \Rightarrow m^2+3 = 0$

$$\Rightarrow m = \sqrt{3}i \text{ or } -\sqrt{3}i$$

$$\Rightarrow y_c = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x$$

$\because$ RHS $= -48x^2e^{3x}$

$$\Rightarrow y_p \triangleq (Ax^2+Bx+c)e^{3x}$$

$$y_p' = (2Ax+B)e^{3x} + (Ax^2+Bx+c)3e^{3x}$$

$$y_p'' = (2A)e^{3x} + (2Ax+B)3e^{3x}$$

$$+3(2Ax+B)e^{3x} + (Ax^2+Bx+c)9e^{3x}$$

$$= e^{3x}(2A+12Ax+6B+9Ax^2+9Bx+9c)$$

$$= e^{3x}[9Ax^2+(12A+9B)x+(2A+6B+9c)]$$

$$y''+3y = e^{3x}[9Ax^2+(12A+9B)x+(2A+6B+9c)]$$

$$+3e^{3x}(Ax^2+Bx+c)$$

$$= e^{3x}[12Ax^2+(12A+12B)x+(2A+6B+12c)]$$

$$= -48x^2 e^{3x}$$

$$\Rightarrow \begin{cases} 12A = -48 & \Rightarrow A = -4 \\ 12A + 12B = 0 & \Rightarrow B = 4 \\ 2A + 6B + 12C = 0 & \Rightarrow C = -\frac{4}{3} \end{cases}$$

$$\Rightarrow y_p = \left(-4x^2 + 4x - \frac{4}{3}\right) e^{3x}$$

$\therefore$

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x \\ &+ \left(-4x^2 + 4x - \frac{4}{3}\right) e^{3x} \end{aligned}}$$

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(4.4) $y'' + 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

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$y \triangleq e^{mx}$ for AHE

$(m^2 e^{mx}) + 2(me^{mx}) + 2(e^{mx}) = 0$

$\Rightarrow e^{mx}[m^2 + 2m + 2] = 0$

$\because e^{mx} \neq 0 \quad \forall x$

$\Rightarrow m^2 + 2m + 2 = 0$

$\Rightarrow m = 1+i$ or $1-i$

$\Rightarrow y_c = e^{x}(c_1\cos x + c_2 \sin x)$

$\because$ RHS $= e^{2x}(\cos x - 3\sin x)$

$y_p \triangleq Ae^{2x}\cos x + Be^{2x}\sin x$

$y_p' = 2Ae^{2x}\cos x - Ae^{2x}\sin x$
$\quad + 2Be^{2x}\sin x + Be^{2x}\cos x$
$\quad = (2A+B)e^{2x}\cos x + (2B-A)e^{2x}\sin x$

$y_p'' = (2A+B)2e^{2x}\cos x - (2A+B)e^{2x}\sin x$
$\quad + (2B-A)2e^{2x}\sin x + (2B-A)e^{2x}\cos x$
$\quad = (3A+4B)e^{2x}\cos x + (-4A+3B)e^{2x}\sin x$

$$y_p'' - 2y_p' + 2y_p$$

$$= e^{2x}\cos x\,(2A - 4A - 2B + 3A + 4B)$$

$$+ e^{2x}\sin x\,(2B - 4B + 2A - 4A + 3B)$$

$$= e^{2x}\cos x\,(A + 2B) + e^{2x}\sin x\,(B - 2A)$$

$$= e^{2x}(\cos x - 3\sin x)$$

$$\Rightarrow \begin{cases} A + 2B = 1 \\ B - 2A = -3 \end{cases} \Rightarrow \begin{cases} A = \frac{7}{5} \\ B = -\frac{1}{5} \end{cases}$$

$$\Rightarrow y_p = \frac{7}{5}e^{2x}\cos x - \frac{1}{5}e^{2x}\sin x$$

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= e^x(c_1\cos x + c_2\sin x) \\ &\quad + \frac{7}{5}e^{2x}\cos x - \frac{1}{5}e^{2x}\sin x \end{aligned}}$$

(4.4) $\frac{d^2x}{dt^2} + w^2 x = F_0 \sin wt$

33 $x(0) = 0, \quad x'(0) = 0$

For AHE $\frac{d^2x}{dt^2} + w^2 x = 0$

$x = e^{mx} \Rightarrow m^2 e^{mx} + w^2 e^{mx} = 0$

$e^{mx} \neq 0, \forall x \Rightarrow m^2 + w^2 = 0 \quad m = \pm wi$

$\Rightarrow x_c = c_1 \cos wt + c_2 \sin wt$

$\because$ RHS $= F_0 \sin wt$,

and has same function in x_c

$\therefore x_p \triangleq At \cos wt + Bt \sin wt$

$x_p' = A\cos wt - Atw\sin wt + B\sin wt + Btw\cos wt$

$= \cos wt (A + Bwt) + \sin wt (B - Awt)$

$x_p'' = (-w\sin wt)(A + Bwt) + \cos wt\,(Bw)$

$+ (w\cos wt)(B - Awt) + \sin wt\,(-Aw)$

$= \sin wt(-2Aw - Bw^2 t) + \cos wt\,(2Bw - Aw^2 t)$

$x_p'' + w^2 x_p = \sin wt(-2Aw - \cancel{Bw^2 t} + \cancel{Bw^2 t})$

$+ \cos wt\,(2Bw - \cancel{Aw^2 t} + \cancel{Aw^2 t})$

$= F_0 \sin wt$

$$\Rightarrow \begin{cases} -2A\omega = F_0 \Rightarrow A = \dfrac{-F_0}{2\omega} \\ B\omega = 0 \Rightarrow B = 0 \end{cases}$$

$$\Rightarrow x_p = -\frac{F_0}{2\omega} t \cos\omega t$$

$$x = x_c + x_p = c_1 \cos\omega t + c_2 \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t$$

$$x(0) = 0, \quad x'(0) = 0$$

$$0 = c_1 - 0 \Rightarrow c_1 = 0$$

$$0 = c_1 \omega \cdot 0 + c_2 \omega \cdot 1 - \frac{F_0}{2\omega} \omega \cdot 1 - 0$$

$$\Rightarrow c_2 = \frac{F_0}{2\omega^2}$$

$$\Rightarrow \boxed{x = \frac{F_0}{2\omega^2} \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t}$$

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$$y'' + 3y = 6x$$

$$y(0)=0, \quad y(1)+y'(1) = 0$$

$$\text{AHE} \Rightarrow y''+3y = 0$$

$$y_c = e^{mx} \Rightarrow m^2 e^{mx} + 3e^{mx} = 0$$

$$\because e^{mx} \neq 0 \ \forall x \Rightarrow m^2+3 = 0 \Rightarrow m = \pm\sqrt{3}\, i$$

$$\Rightarrow y_c = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$$

$$\text{RHS} = 6x$$

$$y_p = Ax+B, \quad y_p' = A, \quad y_p'' = 0$$

$$\Rightarrow 0 + 3(Ax+B) = 6x$$

$$\Rightarrow A = 2, \quad B = 0$$

$$y = y_c + y_p = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + 2x$$

$$y(0)=0 \Rightarrow 0 = c_1$$

$$y(1)+y'(1)=0$$

$$\Rightarrow \left(c_2\sin\sqrt{3}+2\right) + \left(c_2\sqrt{3}\cos\sqrt{3}+2\right) = 0$$

$$c_2\left(\sin\sqrt{3}+\sqrt{3}\cos\sqrt{3}\right) = -4$$

$$c_2 = (-4)\,/\left(\sin\sqrt{3}+\sqrt{3}\cos\sqrt{3}\right)$$

$$\Rightarrow \boxed{y = \frac{-4\sin\sqrt{3}x}{\left(\sin\sqrt{3}+\sqrt{3}\cos\sqrt{3}\right)} + 2x}$$

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