(4.6) $y'' + y = \cos^2 x \Leftrightarrow y'' + Py' + Qy = f(x)$

$y \triangleq e^{mx}$ $\qquad P = 0 \quad Q = 1$

5

From the auxiliary equation

$$m^2 + 1 = 0 \qquad m = i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$f(x) = \cos^2 x$$

$$y_p \triangleq u_1 y_1 + u_2 y_2 \qquad y_1 = \cos x,\ y_2 = \sin x$$

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$y_p'' = \cdots$$

$$y_p'' + P y_p' + Q y_p = f(x)$$

$$\Rightarrow \frac{d}{dx}[y_1 u_1' + y_2 u_2'] + \cancel{P}^{0}[y_1 u_1' + y_2 u_2']$$

$$+ [y_1' u_1' + y_2' u_2'] = f(x)$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

$$W_1 \triangleq \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{vmatrix} = -\sin x \cos^2 x$$

$$W_2 \triangleq \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{vmatrix} = \cos^3 x$$

$$u_1' = \frac{W_1}{W} = -\sin x \cos^2 x, \quad u_2' = \frac{W_2}{W} = \cos^3 x$$

$$\Rightarrow \begin{cases} u_1 = \frac{1}{3}\cos^3 x \\ u_2 = \sin x - \frac{1}{3}\sin^3 x \end{cases}$$

$$\Rightarrow y_p = \frac{1}{3}\cos^3 x \cos x + \left(\sin x - \frac{1}{3}\sin^3 x\right)\sin x$$

$$y = y_c + y_p$$

$$= C_1\cos x + C_2 \sin x + \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x$$

$$= C_1\cos x + C_2 \sin x + \frac{1}{3}\left(\cos^4 x - \sin^4 x\right) + \sin^2 x$$

$$= C_1\cos x + C_2 \sin x + \frac{1}{3}\underbrace{\left(\cos^2 x + \sin^2 x\right)}_{1}\left(\cos^2 x - \sin^2 x\right)$$

$$= C_1\cos x + C_2 \sin x + \frac{1}{3}\cos^2 x - \frac{1}{3}\sin^2 x + \sin^2 x \qquad + \sin^2 x$$

$$= C_1\cos x + C_2 \sin x + \frac{1}{3}\cos^2 x + \frac{2}{3}\sin^2 x \qquad \frac{1}{3}\sin^2 x + \frac{1}{3}\sin^2 x$$

$$= \boxed{C_1\cos x + C_2 \sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x}$$

(14.6) $y''-2y'+y=e^t\tan^{-1}t \Leftrightarrow y''+Py'+Qy=f$

14 $\quad y \triangleq e^{mx}$

The auxiliary equation: $m^2-2m+1=0$

$= (m-1)=0 \Rightarrow m=1,1$

$\Rightarrow y_c \triangleq c_1 e^t + c_2 t e^t \qquad y_1 \triangleq e^t, \quad y_2 \triangleq te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix} = e^{2t}+te^{2t}-te^{2t} = e^{2t}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\Rightarrow \begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f \end{cases}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^t \\ e^t\tan^{-1}t & e^t+te^t \end{vmatrix} = -te^{2t}\tan^{-1}t$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \begin{vmatrix} e^t & 0 \\ e^t & e^t\tan^{-1}t \end{vmatrix} = e^{2t}\tan^{-1}t$$

$$\Rightarrow \begin{cases} u_1' = \dfrac{W_1}{W} = \dfrac{-te^{2t}\tan^{-1}t}{e^{2t}} = -t\tan^{-1}t \\ u_2' = \dfrac{W_2}{W} = \dfrac{e^{2t}\tan^{-1}t}{e^{2t}} = \tan^{-1}t \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -\dfrac{1+t^2}{2}\tan^{-1}t + \dfrac{t}{2} \\ u_2 = t\tan^{-1}t - \dfrac{1}{2}\ln(1+t^2) \end{cases}$$

$$\Rightarrow y = y_c + y_p$$

$$= C_1 e^t + C_2 t e^t$$

$$+ \left(\left(-\frac{1+t^2}{2} \right) \tan^{-1} t + \frac{t}{2} \right) e^t$$

$$+ \left(t \tan^{-1} t - \frac{1}{2} \ln(1+t^2) \right) t e^t$$

$$= C_1 \boxed{e^t} + C_3 \boxed{t e^t} \qquad C_3 = C_2 + \frac{1}{2}$$

$$+ \frac{1}{2} \boxed{e^t} \left[(t^2 - 1) \tan^{-1} t - \ln(1+t^2) \right]$$

(4.6) $y'' + 2y' - 8y = \underbrace{2e^{-2x} - e^{-x}}_{= f(x)}, \quad y(0) = 1,\ y'(0) = 0$

21 $y \triangleq e^{mx}$

the auxiliary equation $= m^2 + 2m - 8 = 0$

$\Rightarrow (m-2)(m+4) = 0 \Rightarrow m = 2, -4$

$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-4x}$ $\quad y_1 \triangleq e^{2x}$, $y_2 \triangleq e^{-4x}$

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}$$

$y_p \triangleq u_1 y_1 + u_2 y_2$

$$\Rightarrow \begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

$$W_1 \triangleq \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{-4x} \\ 2e^{-2x} - e^{-x} & -4e^{-4x} \end{vmatrix}$$

$$= -2e^{-6x} + e^{-5x}$$

$$W_2 \triangleq \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{-2x} - e^{-x} \end{vmatrix}$$

$$= 2 - e^{x}$$

$$\begin{cases} u_1' = \dfrac{W_1}{W} = \dfrac{-2e^{-6x} + e^{-5x}}{-6e^{-2x}} = \dfrac{1}{3}e^{-4x} - \dfrac{1}{6}e^{-3x} \\ u_2' = \dfrac{W_2}{W} = \dfrac{2 - e^{x}}{-6e^{-2x}} = -\dfrac{1}{3}e^{2x} + \dfrac{1}{6}e^{3x} \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -\dfrac{1}{12}e^{-4x} + \dfrac{1}{18}e^{-3x} \\ u_2 = -\dfrac{1}{6}e^{2x} + \dfrac{1}{18}e^{3x} \end{cases}$$

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 e^{-4x}$$

$$+ \left(-\frac{1}{12} e^{-4x} + \frac{1}{18} e^{-3x}\right) e^{2x} + \left(-\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x}\right) e^{-4x}$$

$$= c_1 e^{2x} + c_2 e^{-4x} - \frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x}$$

$$y' = 2c_1 e^{2x} - 4c_2 e^{-4x} + \frac{1}{2} e^{-2x} - \frac{1}{9} e^{-x}$$

$$y(0) = 1 \Rightarrow c_1 + c_2 - \frac{1}{4} + \frac{1}{9} = 1$$

$$y'(0) = 0 \Rightarrow 2c_1 - 4c_2 + \frac{1}{2} - \frac{1}{9} = 0$$

$$\Rightarrow c_1 = \frac{25}{36}, \quad c_2 = \frac{4}{9}$$

$$\Rightarrow \boxed{y = \frac{25}{36} e^{2x} + \frac{4}{9} e^{-4x} - \frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x}}$$

(4.6) 29

$$y''' + y' = \tan x$$

$y \triangleq e^{mx}$ $\qquad f(x)$

The auxiliary equation: $m^3 + m = 0$

$\Rightarrow m(m^2+1) = 0 \Rightarrow m = 0, \pm i$

$\Rightarrow y_c = c_1 + c_2 \cos x + c_3 \sin x$

$y_1 = 1 \qquad y_2 \qquad y_3$

$$\Rightarrow W \triangleq \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1.$$

$$\Rightarrow y_p \triangleq u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$\begin{cases} y_1 u_1' + y_2 u_2' + y_3 u_3' = 0 \\ y_1' u_1' + y_2' u_2' + y_3' u_3' = 0 \\ y_1'' u_1' + y_2'' u_2' + y_3'' u_3' = f \end{cases}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f \end{vmatrix} = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x$$

$$\begin{cases} u_1' = \dfrac{W_1}{W} = \tan x \\ u_2' = \dfrac{W_2}{W} = -\sin x \\ u_3' = \dfrac{W_3}{W} = -\sin x \tan x = \cos x - \sec x \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -\ln|\cos x| \\ u_2 = \cos x \\ u_3 = \sin x - \ln|\sec x + \tan x| \end{cases}$$

$$\Rightarrow y = y_c + y_p$$

$$= C_1 + C_2 \cos x + C_3 \sin x$$

$$- \ln|\cos x| + \cos^2 x + \sin^2 x - \sin x \ln|\sec x + \tan x|$$

$$\boxed{\begin{aligned} &= C_4 + C_2 \cos x + C_3 \sin x \\ &\quad - \ln|\cos x| - \sin x \ln|\sec x + \tan x| \end{aligned}}$$

$C_4 = C_1 + 1$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$