(4.7) $x^2y'' + 5xy' + 4y = 0$

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$y \triangleq x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$\Rightarrow x^2(m\cdot(m-1)x^{m-2}) + 5x(m x^{m-1}) + 4(x^m) = 0$

$m(m-1)x^m + 5mx^m + 4x^m = 0$

$x^m(m(m-1) + 5m + 4) = 0$

$x^m \neq 0$, $\Rightarrow$ the auxiliary equation =

$m(m-1) + 5m + 4 = 0$

$\Rightarrow m^2 - m + 5m + 4 = 0$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow m = -2, -2$ 重根!

$\Rightarrow \boxed{y = c_1 x^{-2} + c_2 x^{-2}(\ln x)}$ #

(4.7) $x^2y'' - xy' + y = 2x$ By variation of parameters

21 $y_c \triangleq x^m$

$y_c' = mx^{m-1}$

$y_c'' = m(m-1)x^{m-2}$

$\Rightarrow$ A+E =

$x^2(m(m-1)x^{m-2}) - x(mx^{m-1}) + x^m = 0$

$x^m(m(m-1) - m + 1) = 0$

$\because x^m \neq 0 \Rightarrow m(m-1) - m + 1 = 0$

$\Rightarrow m^2 - m - m + 1 = 0 \Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\Rightarrow y_c = c_1 x + c_2 x \ln x$

$y_1 \triangleq x \quad , \quad y_2 \triangleq x \ln x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + x \cdot \frac{1}{x} \end{vmatrix}$$

$= x\ln x + x - x\ln x = x$

for $x \neq 0$ or $\underline{\underline{x > 0}}$

$$\Rightarrow \quad y'' \underbrace{- \frac{1}{x}}_{P(x)} y' + \underbrace{\frac{1}{x^2}}_{Q(x)} y = \underset{f(x)}{\overset{\parallel}{\frac{2}{x}}}$$

$$\Rightarrow \quad y_p \triangleq u_1 y_1 + u_2 y_2$$

$$\Rightarrow \begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

$$W_1 \triangleq \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x + 1 \end{vmatrix}$$

$$= -2\ln x$$

$$W_2 \triangleq \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2$$

$$\begin{cases} u_1' = \frac{W_1}{W} = \frac{-2\ln x}{x} \\ u_2' = \frac{W_2}{W} = \frac{2}{x} \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -(\ln x)^2 \\ u_2 = 2\ln x \end{cases}$$

$$y = y_c + y_p$$

$$= c_1 x + c_2 x\ln x$$

$$+ \left(-(\ln x)^2\right) x + (2\ln x)\, x\ln x$$

$$= c_1 x + c_2 x\ln x + x(\ln x)^2, \quad x > 0$$

(4.7) 32

$$x^2y'' - 9xy' + 25y = 0$$

$x = e^t$ or $t = \ln x$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow \cancel{x^2}\left(-\frac{1}{\cancel{x^2}}\frac{dy}{dt} + \frac{1}{\cancel{x^2}}\frac{d^2y}{dt^2}\right) - 9\cancel{x}\left(\frac{1}{\cancel{x}}\frac{dy}{dt}\right) + 25y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y = 0$$

$$y \hat{=} e^{mt}$$

$\Rightarrow$ the auxiliary equation =

$$m^2e^{mt} - 10me^{mt} + 25e^{mt} = 0$$

$$e^{mt}(m^2 - 10m + 25) = 0$$

$$\because e^{mt} \neq 0 \Rightarrow m^2 - 10m + 25 = 0$$

$$\Rightarrow (m-5)^2 = 0 \Rightarrow m = 5, 5 \text{ 重根}$$

$$\boxed{y = c_1e^{5t} + c_2te^{5t}}$$

or $\boxed{y = c_1x^5 + c_2(\ln x)x^5}$ $\#$

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$$(x+3)^2 y'' - 8(x+3)y' + 14y = 0$$

$$y \triangleq (x - x_0)^m$$

$$\Rightarrow y \triangleq (x - (-3))^m = (x+3)^m$$

$$y' = m(x+3)^{m-1}$$

$$y'' = m(m-1)(x+3)^{m-2}$$

$$\Rightarrow (x+3)^2 \left(m(m-1)(x+3)^{m-2}\right) - 8(x+3)m(x+3)^{m-1}$$

$$+14(x+3)^m = 0$$

$$\Rightarrow (x+3)^m \left[m(m-1) - 8m + 14 \right] = 0$$

$$(x+3)^m \left[m^2 - 9m + 14 \right] = 0$$

$$(x+3)^m \left[m-7 \right] \left[m-2 \right] = 0$$

$$\because (x+3)^m \neq 0$$

$\Rightarrow$ the auxiliary equation $= (m-7)(m-2) = 0$

$$\Rightarrow m = 2, 7$$

$$\Rightarrow \boxed{y = c_1 (x+3)^2 + c_2 (x+3)^7}$$