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$$\mathcal{L}\{t e^{-3t} \cos 3t\}$$

Thm 7.4.1

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}\{t e^{-3t} \cos 3t\}$$

$$= (-1) \frac{d}{ds} \mathcal{L}\{e^{-3t} \cos 3t\}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+3^2} \Big|_{s \to s+3} \right]$$

$$= -\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+9} \right]$$

$$= -\frac{1}{[(s+3)^2+9]^2}\left[1((s+3)^2+9) - (s+3)(2(s+3))\right]$$

$$(s+3)(2s+6)$$

$$= -\frac{s^2+6s+9+9-2s^2-12s-18}{[(s+3)^2+9]^2}$$

$$= \frac{s^2+6s}{[(s+3)^2+9]^2}$$

$$= \boxed{\frac{(s+3)^2-9}{[(s+3)^2+9]^2}}$$

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$$y'' + 16y = f(t), \quad y(0)=0, \; y'(0)=1$$

$$f(t) = \begin{cases} \cos 4t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

$$\mathcal{L}\{y'' + 16y\} = \mathcal{L}\{f(t)\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 16\mathcal{L}\{y\}$$

$$= \mathcal{L}\{\underline{\cos 4t\,\mathcal{U}(0)} - \underline{\cos 4t\,\mathcal{U}(t-\pi)}\}$$

$$(s^2+16)\mathcal{L}\{y\} - 1 = \frac{s}{s^2+4^2} - e^{-\pi s}\mathcal{L}\{\cos 4(t+\pi)\}$$

$$= \frac{s}{s^2+16} - e^{-\pi s}\mathcal{L}\{\cos 4t\}$$

$$= \frac{s}{s^2+16} - e^{-\pi s}\frac{s}{s^2+16}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{1}{s^2+16}\left[1 + \frac{s}{s^2+16} - e^{-\pi s}\frac{s}{s^2+16}\right]$$

$$= \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}$$

$$\Rightarrow y = \frac{1}{4}\sin 4t + \frac{1}{2}\frac{1}{4}t\sin 4t - \frac{1}{2}\frac{1}{4}(t-\pi)\sin 4(t-\pi)\,\mathcal{U}(t-\pi)$$

$$\boxed{= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\sin 4(t-\pi)\,\mathcal{U}(t-\pi)}$$

(7.4) $\mathcal{L}\left\{ \int_0^t e^{-\tau} \cos\tau \, d\tau \right\}$

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$$\Rightarrow = \frac{1}{s} \mathcal{L}\{ e^{-t} \cos t \}$$

$$= \frac{1}{s} \mathcal{L}\{ \cos t \} \Big|_{s \to s+1}$$

$$= \frac{1}{s} \frac{s}{s^2+1} \Big|_{s \to s+1}$$

$$= \frac{1}{s} \frac{s+1}{(s+1)^2+1}$$

$$= \boxed{\frac{s+1}{s(s^2+2s+2)}}$$

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$$f(t) = 1 + t - \frac{8}{3}\int_0^t (\tau - t)^3 f(\tau)\,d\tau$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\left\{1 + t - \frac{8}{3}\int_0^t (\tau - t)^3 f(\tau)\,d\tau\right\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{t\} - \frac{8}{3}\mathcal{L}\left\{\int_0^t \underline{(\tau - t)^3}\,\underline{f(\tau)}\,d\tau\right\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\mathcal{L}\left\{\int_0^t (t-\tau)^3 f(\tau)\,d\tau\right\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\mathcal{L}\{t^3\}\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\,\frac{3!}{s^4}\,\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{16}{s^4}\mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{\frac{1}{s} + \frac{1}{s^2}}{1 - \frac{16}{s^4}}$$

$$= \frac{\frac{s+1}{s^2}}{\frac{s^4-16}{s^4}}$$

$$= \frac{s^2(s+1)}{s^4-16}$$

$$= \frac{1}{8}\frac{1}{s+2} + \frac{3}{8}\frac{1}{(s-2)} + \frac{1}{4}\frac{2}{s^2+4} + \frac{1}{2}\frac{s}{s^2+4}$$

$$\Rightarrow \boxed{f(t) = \frac{1}{8}e^{-2t} + \frac{3}{8}e^{2t} + \frac{1}{4}\sin 2t + \frac{1}{2}\cos 2t}$$

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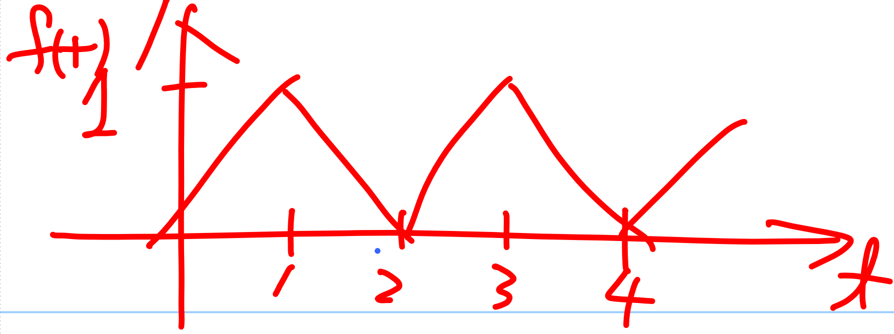

$t \quad (2-t)$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s\cdot 2}}\left[\int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt\right]$$

$$\int_0^1 \underset{u}{t}\,\underset{v'}{e^{-st}}\,dt = t\frac{1}{(-s)}e^{-st}\Big|_0^1 - \int_0^1 \frac{1}{(-s)}e^{-st}\,dt$$

$$= \frac{1}{(-s)}\left[1\cdot e^{-s}-0\right] - \frac{1}{(-s)}\frac{1}{(-s)}e^{-st}\Big|_0^1$$

$$= \frac{-e^{-s}}{s} - \frac{1}{s^2}\left(e^{-s}-1\right)$$

$$\int_1^2 (2-t)e^{-st}\,dt = 2\int_1^2 e^{-st} - \int_1^2 te^{-st}$$

$$= 2\frac{1}{(-s)}\left(e^{-2s}-e^{-s}\right) - \left[\frac{1}{(-s)}\left(2e^{-2s}-1e^{-s}\right) - \frac{1}{s^2}\left(e^{-2s}-e^{-s}\right)\right]$$

$$= \frac{2}{-s}\left(e^{-2s}-e^{-s}\right) + \frac{1}{s}\left(2e^{-2s}-e^{-s}\right) + \frac{1}{s^2}\left(e^{-2s}-e^{-s}\right)$$

$$\Rightarrow \cancel{-\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \cancel{-\frac{2e^{-2s}}{s}} + \cancel{\frac{2e^{-s}}{s}} + \cancel{\frac{2e^{-2s}}{s}} \cancel{-\frac{e^{-s}}{s}} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

$$= -\frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}\left(1+e^{-2s}-2e^{-s}\right) = \frac{1}{s^2}\left(1-e^{-s}\right)^2$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\,\frac{1}{s^2}\left(1-e^{-s}\right)^2$$

$$= \frac{1}{\cancel{(1-e^{-s})}(1+e^{-s})}\,\frac{1}{s^2}\left(1-e^{-s}\right)^{\cancel{2}}$$

$$= \boxed{\frac{1-e^{-s}}{s^2\left(1+e^{-s}\right)}}$$

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