

Fall 2007

# 線性系統 Linear Systems

## Chapter 02 Mathematical Descriptions of Systems

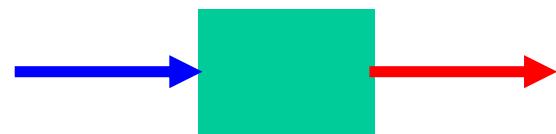
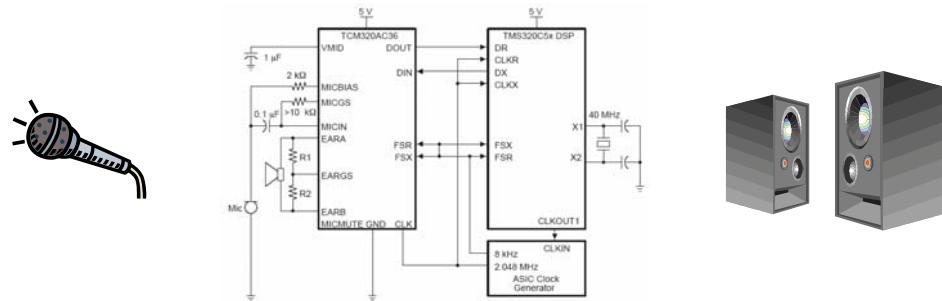
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Sep07 – Jan08

Materials used in these lecture notes are adopted from  
“Linear System Theory & Design,” 3rd. Ed., by C.-T. Chen (1999)

### Outline

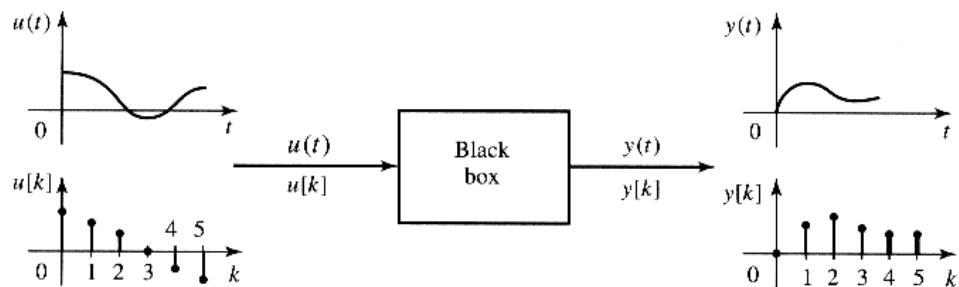
- Introduction
- Linear Systems (2.2)
- Linear Time-Invariant Systems (2.3)
- Linearization (2.4)
- Examples (2.5)
- Discrete-Time Systems (2.6)

## Introduction: What is a system? – 1 (2.1)



## Introduction: What is a system? – 2

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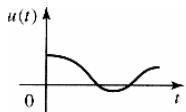
**Figure 2.1** System.

Excitation  
Input  
Cause  
**System**  
Response  
Output  
Effect

$$\left. \begin{array}{l} x(t_0) \\ u(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow y(t), \quad t \geq t_0$$

→ A “mapping” from **input** signal(s) to **output** signal(s)

- Continuous-Time systems:



$$\Rightarrow u(t), y(t)$$

- Discrete-Time systems:



$$\Rightarrow \begin{cases} u[k] = u(t = kT) \\ y[k] = y(t = kT) \end{cases}$$

- Single-Input-Single-Output (SISO) systems:



$$\Rightarrow u(t), y(t)$$

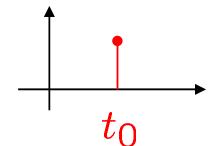
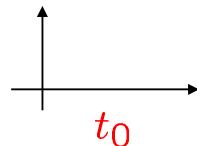
- Multi-Input-Multi-Output (MIMO) systems:



$$\Rightarrow \mathbf{u}(t) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

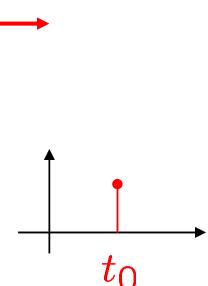
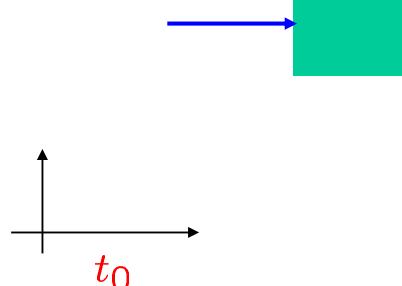
- Memoryless systems:

➤  $y(t_0)$  depends only on  $u(t_0)$



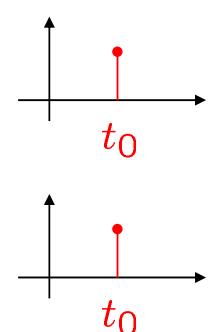
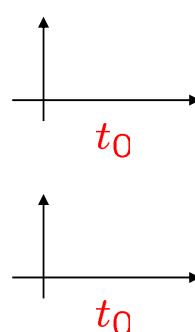
- Dynamic systems:

➤  $y(t_0)$  depends on  $u(t)$



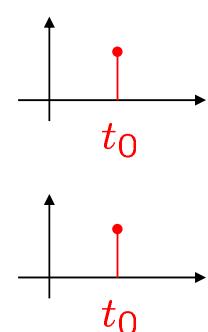
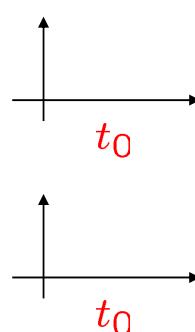
- Causal systems:

➤  $y(t_0)$  depends only on  $u(t), t \leq t_0$



- Non-causal (anticipatory) systems:

➤  $y(t_0)$  depends on  $u(t), t \geq t_0$

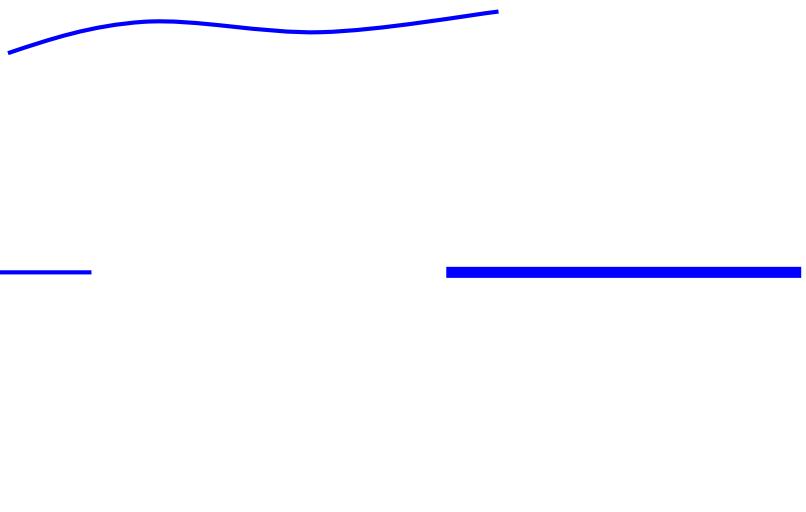


- **Lumped systems:**

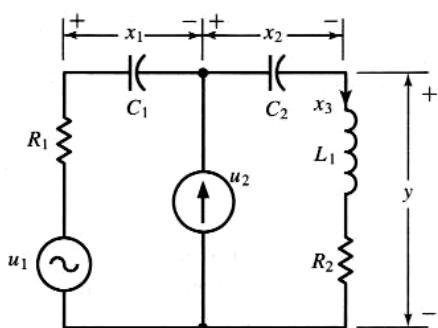
- Finite number of state variables

- **Distributed systems:**

- Infinite number of state variables



- System Description



- Inputs:

- Outputs:

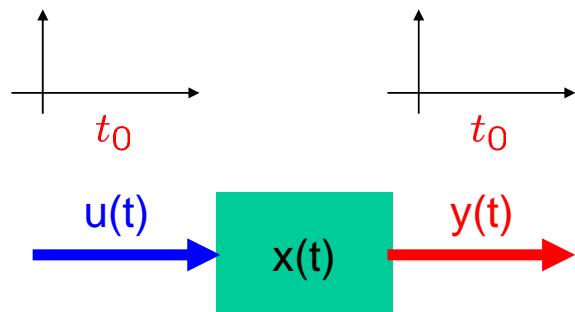
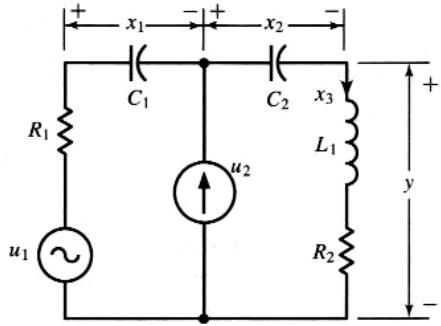
- Parameters:

- Variables:

- State Variables:

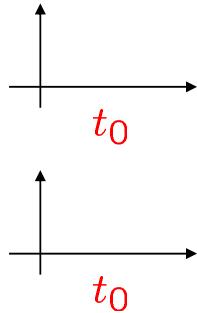
- Definition 2.1: State

The state  $\mathbf{x}(t_0)$  of a system at time  $t_0$  is the information at  $t_0$  that, together with the input  $\mathbf{u}(t)$ , for  $t \geq t_0$ . determines uniquely the output  $\mathbf{y}(t)$ ,  $\forall t \geq t_0$ .

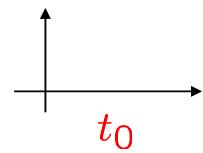


$$\mathbf{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix}$$

$$\left. \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}(t), \quad t \geq t_0$$



$$\left. \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}(t), \quad t \geq t_0$$



- The output is partly excited by the initial state at  $t_0$  and partly by the input applied at and after  $t_0$ .
- Therefore, there is no need to know the input applied before  $t_0$  all the way back to  $-\infty$
- In general, state variables may or may not have physical meanings.

## Linear Systems – 1 (2.2)

- Linear systems satisfy the superposition property, that is,

$$\left. \begin{array}{l} \mathbf{x}_1(t_0) \\ \mathbf{u}_1(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}_1(t), \quad t \geq t_0$$

$$\left. \begin{array}{l} \mathbf{x}_2(t_0) \\ \mathbf{u}_2(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}_2(t), \quad t \geq t_0$$

- Imply (1) additivity

$$\left. \begin{array}{l} \mathbf{x}_1(t_0) + \mathbf{x}_2(t_0) \\ \mathbf{u}_1(t) + \mathbf{u}_2(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}_1(t) + \mathbf{y}_2(t), \quad t \geq t_0$$

- And (2) homogeneity

$$\left. \begin{array}{l} a \cdot \mathbf{x}_1(t_0) \\ a \cdot \mathbf{u}_1(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow a \cdot \mathbf{y}_1(t), \quad t \geq t_0$$

## Linear Systems – 2

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- Zero-input response:  $\left. \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t) \equiv \mathbf{0}, \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}_{zi}(t), \quad t \geq t_0$

- Zero-state response:  $\left. \begin{array}{l} \mathbf{x}(t_0) = \mathbf{0} \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}_{zs}(t), \quad t \geq t_0$

- The additivity property implies:

$$\left. \begin{array}{l} \mathbf{x}(t_0) = \mathbf{x}(t_0) + \mathbf{0} \\ \mathbf{u}(t) = \mathbf{0} + \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}(t) = \mathbf{y}_{zi}(t) + \mathbf{y}_{zs}(t), \quad t \geq t_0$$

Output due to  $\left\{ \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\}$  = output due to  $\left\{ \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t) \equiv \mathbf{0}, \quad t \geq t_0 \end{array} \right\}$  + output due to  $\left\{ \begin{array}{l} \mathbf{x}(t_0) = \mathbf{0} \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\}$

$\therefore$  total response = zero-input response + zero-state response

- Suppose for an SISO linear system,

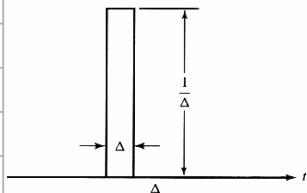
single-input-single-output

if the input is

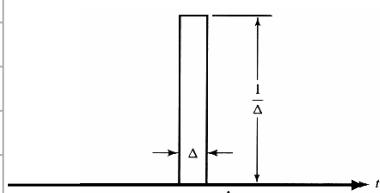
$$u(t) = \delta_{\Delta}(t - t_i)$$

an output is assumed to be

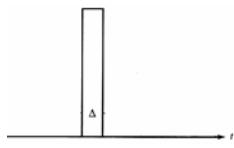
$$y_{zs}(t) = g_{\Delta}(t, t_i)$$



$$\delta_{\Delta}(t - t_i) \longrightarrow g_{\Delta}(t, t_i)$$



$$\delta_{\Delta}(t - t_i) \longrightarrow \text{System} \longrightarrow g_{\Delta}(t, t_i)$$



$$\delta_{\Delta}(t - t_i) \longrightarrow g_{\Delta}(t, t_i)$$

$$\delta_{\Delta}(t - t_{i+1})$$

$$\longrightarrow \text{System} \longrightarrow g_{\Delta}(t, t_{i+1})$$

$$\delta_{\Delta}(t - t_{i+2})$$

$$\longrightarrow \text{System} \longrightarrow g_{\Delta}(t, t_{i+2})$$

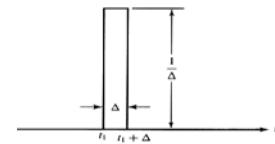
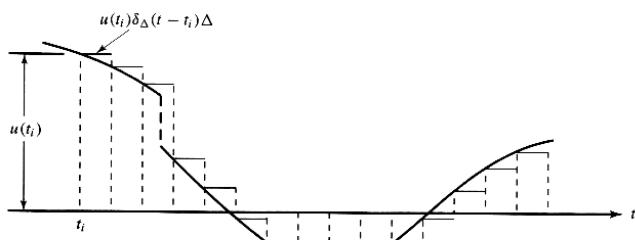
$$\delta_{\Delta}(t - t_{i+3})$$

$$\longrightarrow \text{System} \longrightarrow g_{\Delta}(t, t_{i+3})$$

$$\delta_{\Delta}(t - t_{i+4})$$

$$\longrightarrow \text{System} \longrightarrow g_{\Delta}(t, t_{i+4})$$

- Then for  $u(t)$  approximated by



$$u(t) \approx$$

- Because

$$\delta_Delta(t - t_i) \rightarrow g_Delta(t, t_i)$$

- We have

$$\delta_Delta(t - t_i) u(t_i) \Delta \rightarrow$$

$$\sum_i \delta_Delta(t - t_i) u(t_i) \Delta \rightarrow$$

- Thus

$$y(t) \approx$$

- Let  $\lim_{\Delta \rightarrow 0} \delta_\Delta(t - t_i) = \delta(t - t_i)$  (time-shifted unit-impulse)

- If  $\lim_{\Delta \rightarrow 0} g_\Delta(t, t_i) = g(t, t_i)$  exists, then we have

$$y(t) =$$

- For causal systems,  $g(t, \tau) = 0$ , for  $t < \tau$

And, if the system is relaxed at  $t_0$ , i.e.,  $x(t_0) = 0$

$$y(t) =$$

## The Impulse Response Matrix

- For MIMO linear systems,

Multiple-Input-Multiple-Output

$$\mathbf{y}(t) =$$

where  $\mathbf{G}(t, \tau) = \begin{bmatrix} g_{11}(t, \tau) & g_{12}(t, \tau) & \cdots & g_{1p}(t, \tau) \\ g_{21}(t, \tau) & g_{22}(t, \tau) & \cdots & g_{2p}(t, \tau) \\ \vdots & \vdots & & \vdots \\ g_{q1}(t, \tau) & g_{q2}(t, \tau) & \cdots & g_{qp}(t, \tau) \end{bmatrix}$

is the impulse response matrix

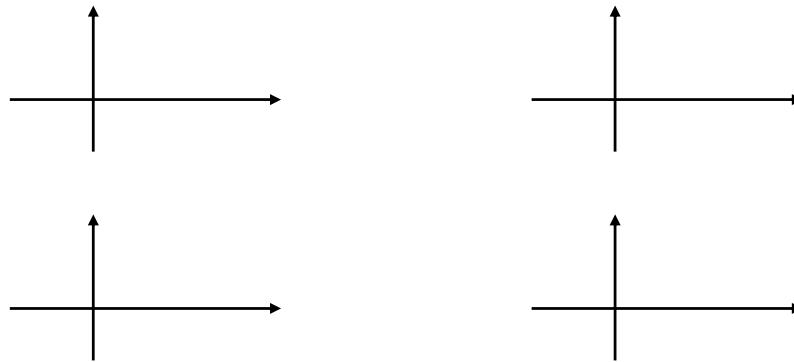
- The  $n$ -dimensional State-Space model,

- $$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$$

where  $\mathbf{y}: q \times 1$ ,  $\mathbf{x}: n \times 1$ ,  $\mathbf{u}: p \times 1$ ,  $\mathbf{A}: n \times n$ ,  $\mathbf{B}: n \times p$ ,  $\mathbf{C}: q \times n$ , and  $\mathbf{D}: q \times p$ .

### Linear Time-Invariant (LTI) Systems – 1 (2.3)

- A system is said to be time invariant



- If for every state-input-output pair

$$\left. \begin{array}{l} \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \rightarrow \mathbf{y}(t), \quad t \geq t_0$$

- and any  $T$ , we have

$$\left. \begin{array}{l} \mathbf{x}(t_0 + T) = \mathbf{x}_0 \\ \mathbf{u}(t - T), \quad t \geq t_0 + T \end{array} \right\} \rightarrow \mathbf{y}(t - T), \quad t \geq t_0 + T$$

(time shifting)

- Since for LTI systems

$$g(t, \tau) = g(t - \tau, 0) = g(t, 0)$$

- We write  $g(t, \tau) = g(t + T, \tau + T) = g(t - \tau, 0) =$

and

$$y(t) =$$

- For SISO LTI systems,  
the impulse response  $g(t)$  and its Laplace transform  $\hat{g}(s)$   
are used to represent the system characteristics

## Transfer Function

$$\hat{y}(s) = \int_0^\infty y(t) e^{-st} dt$$

- Transfer Function Matrix  $\hat{\mathbf{G}}(s)$

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \\ \vdots \\ \hat{y}_q(s) \end{bmatrix} = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \\ \vdots \\ \hat{u}_p(s) \end{bmatrix}$$

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

- Rational transfer function  $\hat{g}(s) = \frac{N(s)}{D(s)}$ ,

where  $N(s), D(s)$  are polynomials for **lumped LTI** syst

- $\hat{g}(s)$  is **proper**  $\Leftrightarrow \deg D(s) \geq \deg N(s)$   
 $\Leftrightarrow \hat{g}(\infty) = 0$  or nonzero constant
- $\hat{g}(s)$  is **strictly proper**  $\Leftrightarrow \deg D(s) > \deg N(s)$   
 $\Leftrightarrow \hat{g}(\infty) = 0$
- $\hat{g}(s)$  is **biproper**  $\Leftrightarrow \deg D(s) = \deg N(s)$   
 $\Leftrightarrow \hat{g}(\infty) = \text{nonzero constant}$
- $\hat{g}(s)$  is **improper**  $\Leftrightarrow \deg D(s) < \deg N(s)$   
 $\Leftrightarrow |\hat{g}(\infty)| = \infty$

- Poles and zeros of SISO LTI systems

pole  $p_i$  :  $\hat{g}(p_i) = \infty$   $D(p_i) = 0$

zero  $z_j$  :  $\hat{g}(z_j) = 0$   $N(z_j) = 0$

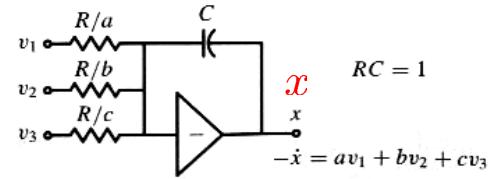
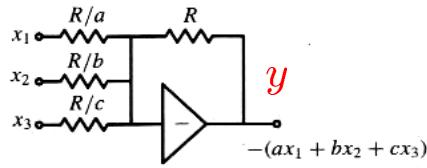
**pole-zero-gain form**  $\hat{g}(s) = k \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$

- Poles and zeros of MIMO LTI systems are more complex

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

## ● Building block



$$\frac{x_1 - 0}{R/a} + \frac{x_2 - 0}{R/b} + \frac{x_3 - 0}{R/c} + \frac{y}{R} = 0$$

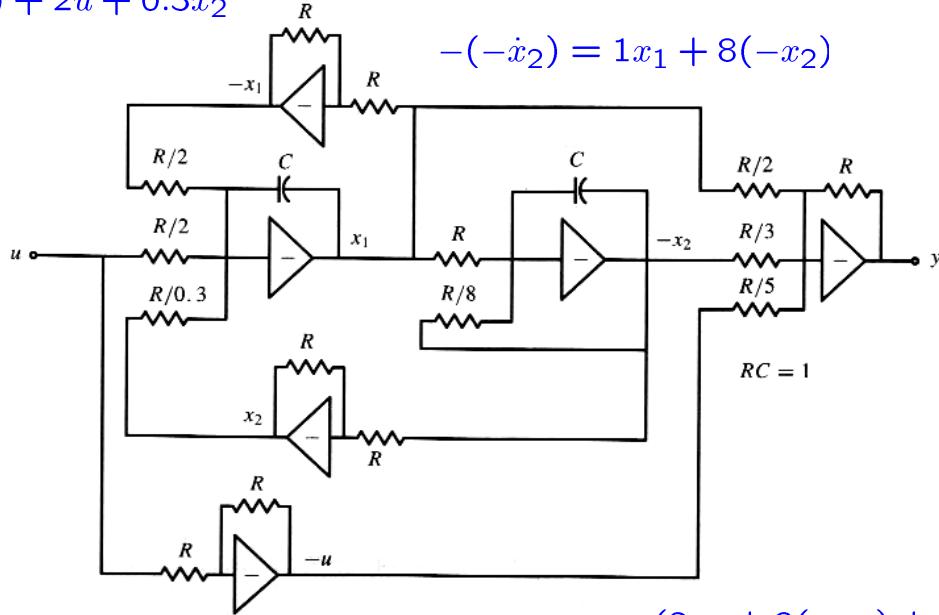
$$\frac{v_1}{R/a} + \frac{v_2}{R/b} + \frac{v_3}{R/c} + C \frac{dx}{dt} = 0$$

$$y = -(ax_1 + bx_2 + cx_3)$$

$$-RC\dot{x} = av_1 + bv_2 + cv_3$$

$$-\dot{x}_1 = 2(-x_1) + 2u + 0.3x_2$$

$$-(-\dot{x}_2) = 1x_1 + 8(-x_2)$$



$$y = -(2x_1 + 3(-x_2) + 5(-u))$$

$$\begin{cases} \dot{x}_1 = 2x_1 - 0.3x_2 - 2u \\ \dot{x}_2 = x_1 - 8x_2 \\ y = -2x_1 + 3x_2 + 5u \end{cases}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -0.3 \\ 1 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [-2 \ 3] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 5u(t)$$

## Linearization – 1 (2.4)

- Linearized state-space model for nonlinear time-varying systems

- Given 
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

- And a set of “nominal solution”  $\{\mathbf{u}_o(t), \mathbf{x}_o(t), \mathbf{y}_o(t)\}$ ,

- How to use a linear state-space model to “approximately represent its characteristics around the nominal solution”

## Linearization – 2

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- For  $\dot{\mathbf{x}}_0 = \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)$ 
  - if  $\mathbf{u}_0(t) \rightarrow \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)$
  - &  $\mathbf{x}_0(0) \rightarrow \mathbf{x}_0(0) + \bar{\mathbf{x}}_0(0)$ ,
  - then  $\mathbf{x}_0(t) \rightarrow \mathbf{x}_0(t) + \bar{\mathbf{x}}(t)$

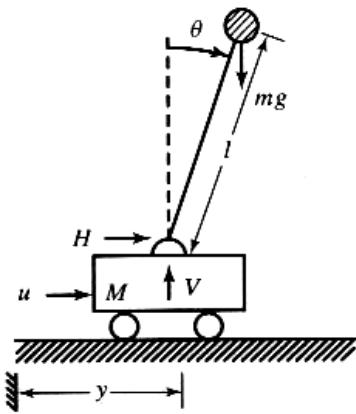
- What is the relationship between  $\bar{\mathbf{u}}(t)$  &  $\bar{\mathbf{x}}(t)$ ?

$$\dot{\mathbf{x}}(t) = \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))$$

$$\dot{\mathbf{x}}_0(t) + \bar{\dot{\mathbf{x}}}(t) = \mathbf{h}(t, \mathbf{x}_0(t) + \bar{\mathbf{x}}(t), \mathbf{u}_0(t) + \bar{\mathbf{u}}(t))$$

$$= \mathbf{h}(t, \mathbf{x}_0(t), \mathbf{u}_0(t)) + \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right|_{\mathbf{x}_0, \mathbf{u}_0} \bar{\mathbf{x}} + \left. \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right|_{\mathbf{x}_0, \mathbf{u}_0} \bar{\mathbf{u}} + \dots$$

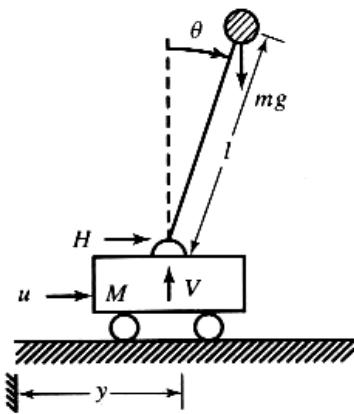
### Example 2.8: Inverted Pendulum on a Cart – 1 (2.5)



Horizontal force balance (linear motion):

Vertical force balance (linear motion):

Torque balance (rotational motion):



Horizontal force balance (linear motion):

$$M \frac{d^2y}{dt^2} = u - H$$

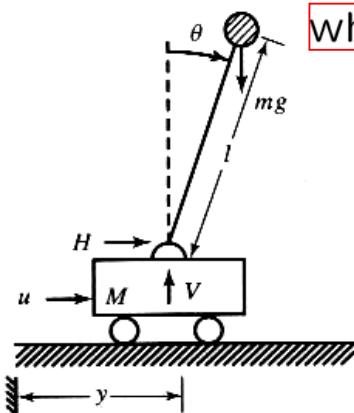
$$H = m \frac{d^2}{dt^2}(y + l \sin \theta) = m\ddot{y} + ml\ddot{\theta} \cos \theta - ml(\dot{\theta})^2 \sin \theta$$

Vertical force balance (linear motion):

$$mg - V = m \frac{d^2}{dt^2}(l \cos \theta) = ml[-\ddot{\theta} \sin \theta - (\dot{\theta})^2 \cos \theta]$$

Torque balance (rotational motion):

$$mgl \sin \theta = ml\ddot{\theta} \cdot l + m\ddot{y}l \cos \theta$$

when  $\theta$ ,  $\dot{\theta}$ , and  $\ddot{\theta}$  are small

Horizontal force balance (linear motion):

$$M \frac{d^2y}{dt^2} = u - H$$

$$H = m \frac{d^2}{dt^2}(y + l \sin \theta) = m\ddot{y} + ml\ddot{\theta} \cos \theta - ml(\dot{\theta})^2 \sin \theta$$

Vertical force balance (linear motion):

$$mg - V = m \frac{d^2}{dt^2}(l \cos \theta) = ml[-\ddot{\theta} \sin \theta - (\dot{\theta})^2 \cos \theta]$$

Torque balance (rotational motion):

$$mgl \sin \theta = ml\ddot{\theta} \cdot l + m\ddot{y}l \cos \theta$$

- Under the “small movement” assumption, the system model is

$$M\ddot{y} = u - m\ddot{y} - ml\ddot{\theta}$$

$$g\theta = l\ddot{\theta} + \ddot{y}$$



$$M\ddot{y} = u - mg\theta$$

$$Ml\ddot{\theta} = (M+m)g\theta - u$$



$$Ms^2\hat{y}(s) = \hat{u}(s) - mg\hat{\theta}(s)$$

$$Mls^2\hat{\theta}(s) = (M+m)g\hat{\theta}(s) - \hat{u}(s)$$



$$\hat{g}_{yu}(s) = \frac{s^2 - g}{s^2[Ms^2 - (M+m)g]}$$

$$\hat{g}_{\theta u}(s) = \frac{-1}{Ms^2 - (M+m)g}$$

- May select a set of state variables as follows:

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \theta$$

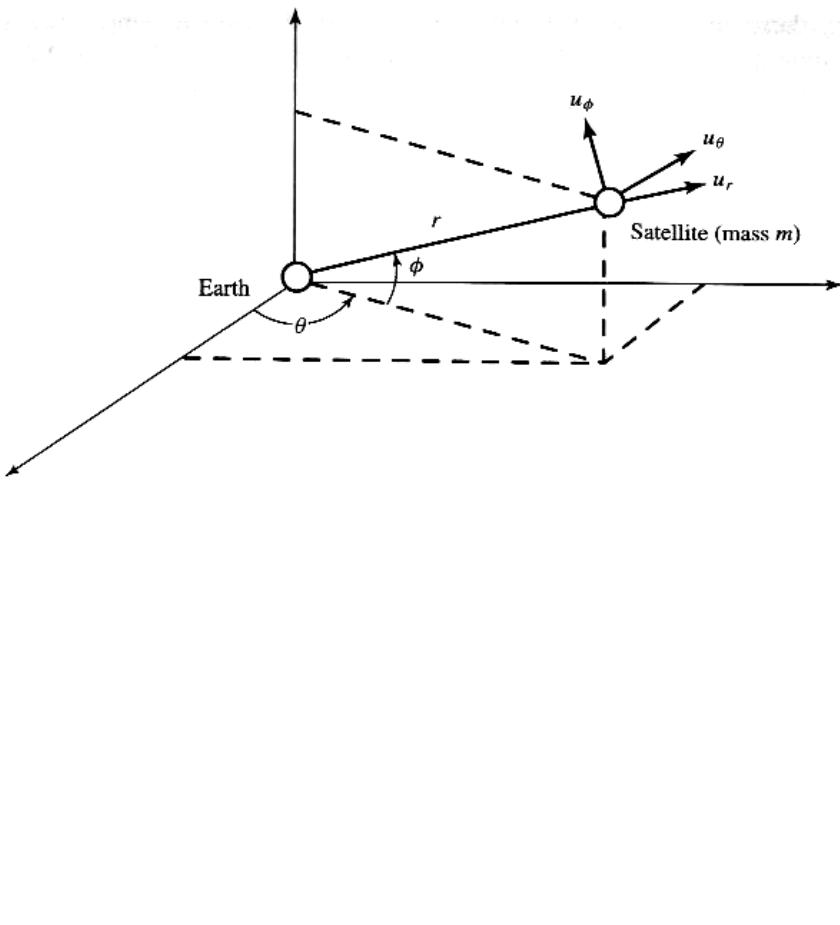
$$x_4 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (M+m)g/Ml & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix} u$$

$y = [1 \ 0 \ 0 \ 0] \mathbf{x}$

Actually a linearized state space model

## Example 2.9: Orbital Movement of a Satellite – 1

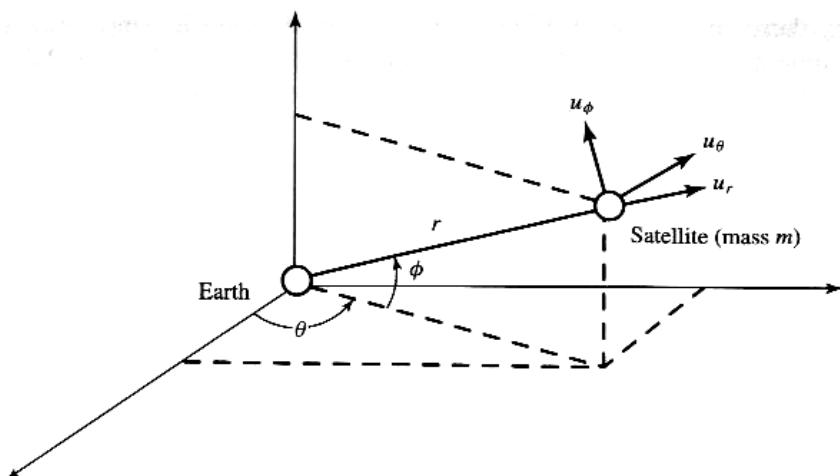


$$\mathbf{x}(t) = \begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_r(t) \\ u_\theta(t) \\ u_\phi(t) \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} r(t) \\ \theta(t) \\ \phi(t) \end{bmatrix}$$

## Example 2.9: Orbital Movement of a Satellite – 2



$$\mathbf{x}(t) = \begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_r(t) \\ u_\theta(t) \\ u_\phi(t) \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} r(t) \\ \theta(t) \\ \phi(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \\ \dot{\phi}(t) \\ \ddot{\phi}(t) \end{bmatrix} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 \cos^2 \phi + r\dot{\phi}^2 - k/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi} \sin \phi / \cos \phi + u_\theta / mr \cos \phi \\ \dot{\phi} \\ -\dot{\theta}^2 \cos \phi \sin \phi - 2\dot{r}\dot{\phi}/r + u_\phi / mr \end{bmatrix} = \mathbf{h}(\mathbf{x}, \mathbf{u})$$

- Nominal solution: circular equatorial constant speed motion

$$\mathbf{x}_o(t) = [r_o \ 0 \ \omega_o t \ \omega_o \ 0 \ 0]' \quad \mathbf{u}_o \equiv \mathbf{0}$$

with  $r_o^3\omega_o^2 = k$ , a known physical constant

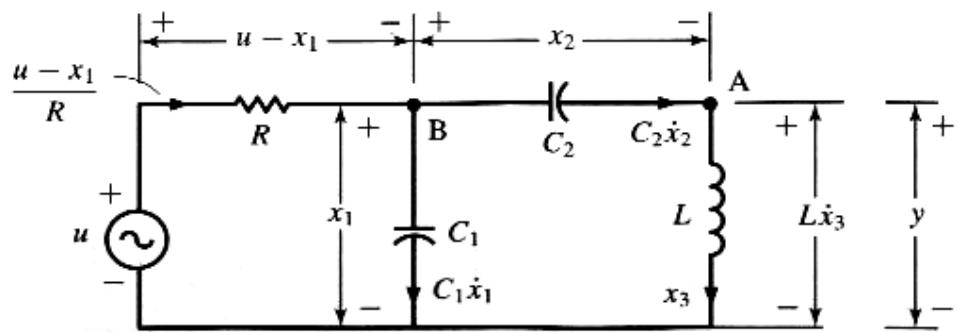
- Linearized model w.r.t. the above nominal solution

- Linearized model w.r.t. the above nominal solution

$$\dot{\bar{\mathbf{x}}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 3\omega_o^2 & 0 & 0 & 2\omega_o r_o & \vdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ 0 & -\frac{2\omega_o}{r_o} & 0 & 0 & \vdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \vdots & -\omega_o^2 & 0 \end{bmatrix} \bar{\mathbf{x}}(t) + \begin{bmatrix} 0 & 0 & \vdots & 0 \\ \frac{1}{m} & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \\ 0 & \frac{1}{mr_o} & \vdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & \frac{1}{mr_o} \end{bmatrix} \bar{\mathbf{u}}(t)$$

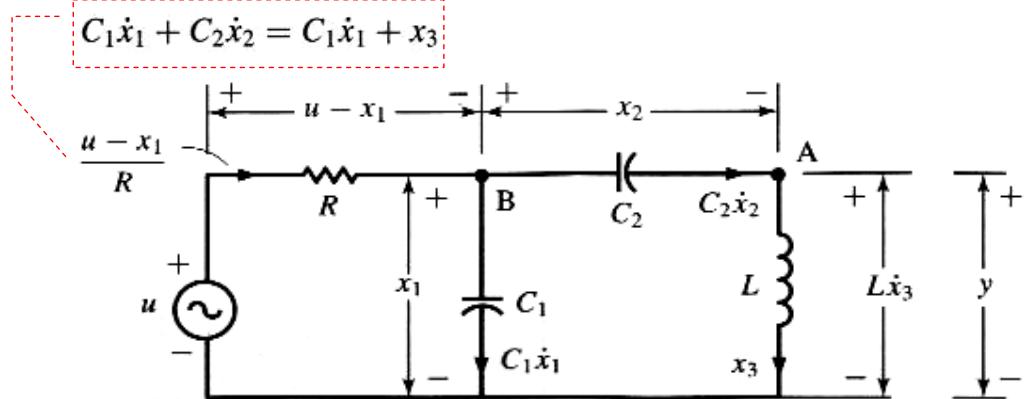
## Example 2.11: An RLC Circuit – 1

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NTUEE-LS2-System-41



## Example 2.11: An RLC Circuit – 2

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NTUEE-LS2-System-42



$$\dot{x}_1 = -\frac{x_1}{RC_1} - \frac{x_3}{C_1} + \frac{u}{RC_1}$$

$$\dot{x}_2 = \frac{1}{C_2}x_3$$

$$\dot{x}_3 = \frac{x_1 - x_2}{L}$$

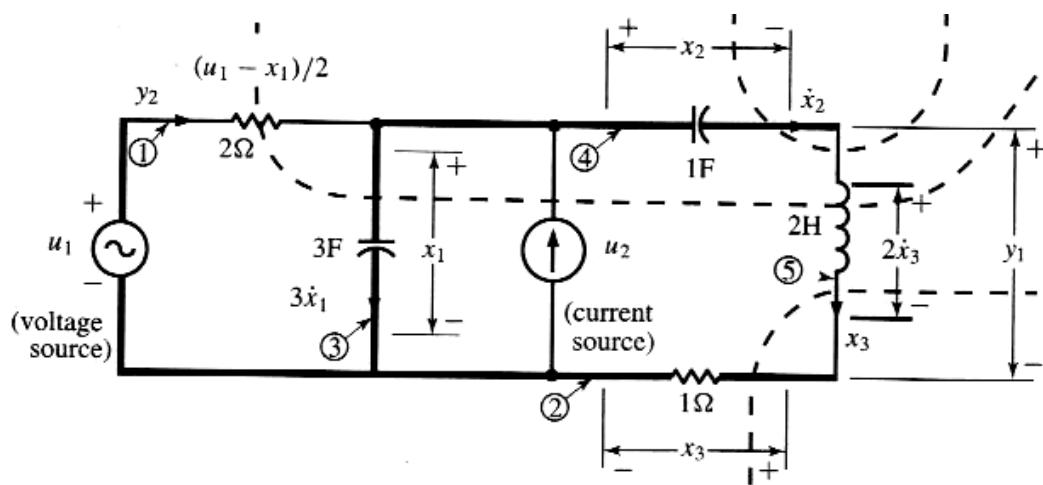
$$y = L\dot{x}_3 = x_1 - x_2$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \ -1 \ 0] \mathbf{x} + 0 \cdot u$$

## Example 2.12: Another RLC Circuit – 1

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NTUEE-LS2-System-43



## Example 2.12: Another RLC Circuit – 2

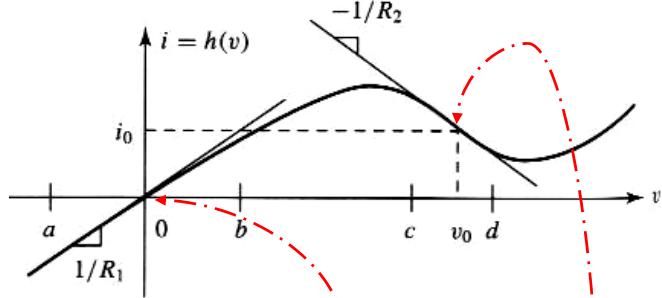
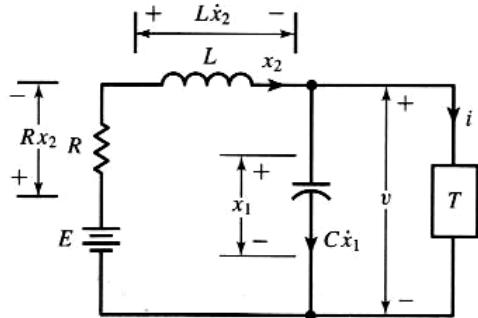
Feng-Li Lian © 2007  
NTUEE-LS2-System-44

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & -1 & -1 \\ -0.5 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} \mathbf{u}$$

## Example 2.12: An RLC Circuit with a Tunnel Diode – 1

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NTUEE-LS2-System-45



$$x_2(t) = C\dot{x}_1(t) + i(t) = C\dot{x}_1(t) + h(x_1(t))$$

$$L\dot{x}_2(t) = E - Rx_2(t) - x_1(t)$$

or, equivalently,

$$\dot{x}_1(t) = \frac{-h(x_1(t))}{C} + \frac{x_2(t)}{C} = H_1(x_1, x_2)$$

$$\dot{x}_2(t) = \frac{-x_1(t) - Rx_2(t)}{L} + \frac{E}{L} = H_2(x_1, x_2, E)$$

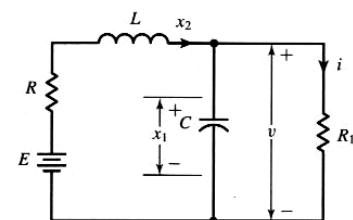
nominal solutions ①, ②,  
decided by  $E$  and  $R$ .

## Example 2.12: An RLC Circuit with a Tunnel Diode – 2

Feng-Li Lian © 2007  
NTUEE-LS2-System-46

- Linearization w.r.t. nominal solution ①:  $x_1 = v = 0, x_2 = i = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/CR_1 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} E$$

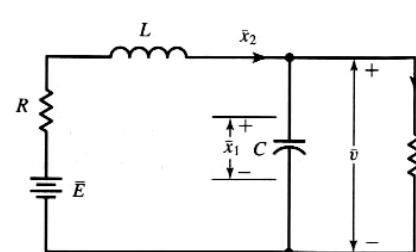


- Linearization w.r.t. nominal solution ②:  $x_1 = v = v_0, x_2 = i = i_0$

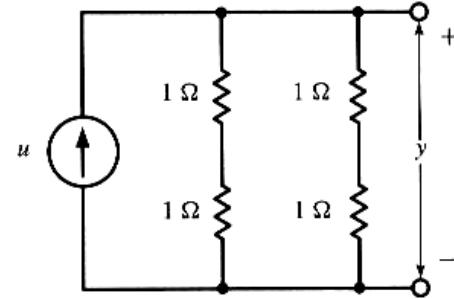
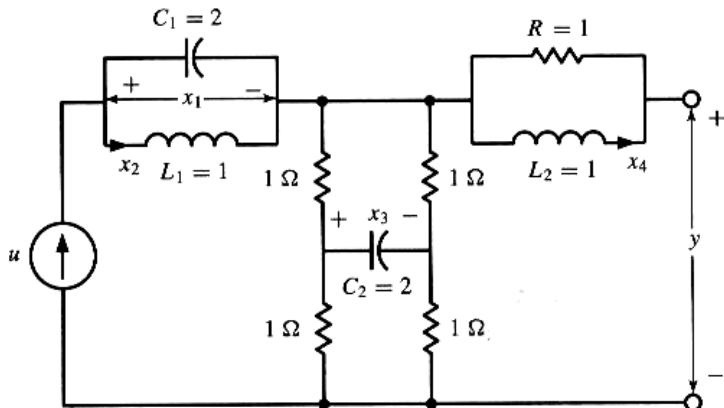
$$\bar{x}_1(t) = x_1(t) - v_0, \quad \bar{x}_2(t) = x_2(t) - i_0$$

$$\bar{E} = E - v_0 - Ri_0$$

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 1/CR_2 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \bar{E}$$



### Example 6.9: Controllable & Observable States – 1



$$y = u$$

$$\hat{g}(s) = 1$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 0 \ 0 \ 1] \mathbf{x} + u$$

$$\dot{\mathbf{x}}_c = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 0] \mathbf{x}_c + u$$

### Summary: A System



$$\left. \begin{array}{l} \mathbf{x}(t_0) \\ \mathbf{u}(t), \quad t \geq t_0 \end{array} \right\} \longrightarrow \mathbf{y}(t), \quad t \geq t_0$$

•  $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$

•  $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t) \\ \mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t) \end{cases} \quad \hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s) \hat{\mathbf{u}}(s)$

•  $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad \hat{g}_{ij}(s) = \frac{N_{ij}(s)}{D_{ij}(s)}$

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & -1 & -1 \\ -0.5 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} \mathbf{u}$$

```
a=[-1/6 0 -1/3;0 0 1;0.5 -0.5 -0.5];
b=[1/6 1/3;0 0;0 0];
c=[1 -1 -1;-0.5 0 0];
d=[0 0;0.5 0];
```

$$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

[N1, d1] = ss2tf(a, b, c, d, 1)

first input

N1 =

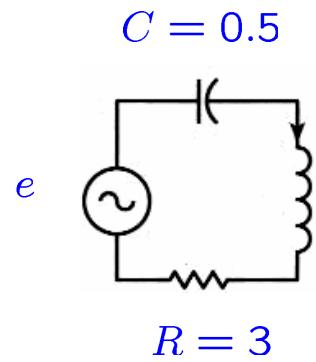
$$\begin{bmatrix} 0.0000 & 0.1667 & -0.0000 & -0.0000 \\ 0.5000 & 0.2500 & 0.3333 & -0.0000 \end{bmatrix}$$

d1 =

$$\begin{bmatrix} 1.0000 & 0.6667 & 0.7500 & 0.0833 \end{bmatrix}$$

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{0.1667s^2}{s^3 + 0.6667s^2 + 0.75s + 0.083} & \frac{0.3333s^2}{s^3 + 0.6667s^2 + 0.75s + 0.0833} \\ \frac{0.5s^3 + 0.25s^2 + 0.3333s}{s^3 + 0.6667s^2 + 0.75s + 0.083} & \frac{-0.1667s^2 - 0.0833s - 0.0833}{s^3 + 0.6667s^2 + 0.75s + 0.083} \end{bmatrix}$$

## Summary: A Simple Example – 1



$$q(t)$$

$$i(t) = \frac{dq(t)}{dt}$$

$$L = 1$$

$$L \frac{dq^2}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = e$$

$$\frac{dq^2}{dt^2} + 3 \frac{dq}{dt} + 2q = e$$

•  $\begin{cases} y &= i \\ e &= 5u \end{cases}$

$$\frac{\hat{y}(s)}{\hat{u}(s)} =$$

## Summary: A Simple Example – 2

$$\frac{dq^2}{dt^2} + 3 \frac{dq}{dt} + 2q = e$$

$$i(t) = \frac{dq(t)}{dt}$$

•  $\begin{cases} x_1 &= q \\ x_2 &= \dot{q} \end{cases}$

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$y =$$

•  $\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

- Discrete-time systems

➤ With a fixed sampling period  $T$  and the integer index  $k$

$$y(kT) =: y[k], \quad u(kT) =: u[k]$$

Concepts like causality, state, lumpedness, linearity, time-invariance, etc., are similar to those in continuous-time systems, also

$$\text{Response} = \text{zero-state response} + \text{zero-input response}$$

- Input-output description: the impulse response model

the impulse sequence  $\delta[k] = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$

arbitrary input  $u[k]$  may be expressed as  $\sum_{m=-\infty}^{\infty} u[m] \delta[k-m]$

$$\delta[k - m] \rightarrow \text{System} \rightarrow g[k, m]$$

- Suppose the input  $\delta[k - m]$  produces  $y_{zs}[k] = g[k, m]$ , then

$$\delta[k - m]u[m] \rightarrow g[k, m]u[m] \quad (\text{homogeneity})$$

$$\sum_m \delta[k - m]u[m] \rightarrow \sum_m g[k, m]u[m] \quad (\text{additivity})$$

thus  $y[k] = \sum_{m=-\infty}^{\infty} g[k, m]u[m]$

impulse response sequence

$$\text{Causal} \iff g[k, m] = 0, \text{ for } k < m$$

$$\text{Time-invariant} \iff y[k] = \sum_{m=0}^k g[k - m]u[m] = \sum_{m=0}^k g[m]u[k - m]$$

discrete convolution

**► The z-transform**

$$\hat{y}(z) := \mathcal{Z}[y[k]] := \sum_{k=0}^{\infty} y[k]z^{-k}$$

**► Discrete transfer function model**

$$\begin{aligned}\hat{y}(z) &= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} g[k-m]u[m] \right) z^{-(k-m)} z^{-m} \\ &= \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} g[k-m]z^{-(k-m)} \right) u[m] z^{-m} \\ &= \left( \sum_{l=0}^{\infty} g[l]z^{-l} \right) \left( \sum_{m=0}^{\infty} u[m]z^{-m} \right) =: \hat{g}(z)\hat{u}(z)\end{aligned}$$

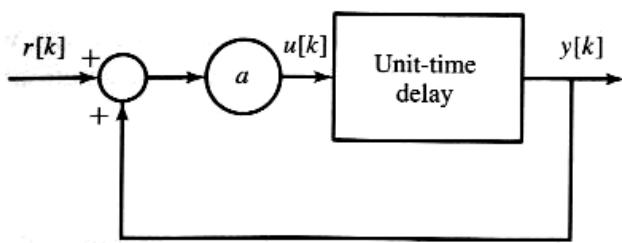
**Examples – 1**

- **Example 2.14:** the unit-sample-time delay system

$$y[k] = u[k - 1]$$

$$\hat{g}(z) = \mathcal{Z}[\delta[k - 1]] = z^{-1} = \frac{1}{z}$$

- Example 2.15: a discrete-time feedback system



transfer function from  $r[k]$  to  $y[k]$

$$\hat{g}(z) = \frac{az^{-1}}{1 - az^{-1}} = \frac{a}{z - a}$$

- Example: a discrete-time system with an irrational  $\hat{g}(z)$

$$g[k] = \begin{cases} 0 & \text{for } m \leq 0 \\ 1/k & \text{for } k = 1, 2, \dots \end{cases}$$

$$\hat{g}(z) = z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} + \dots = -\ln(1 - z^{-1})$$

not a lumped system

- Improper discrete T.F. represent non-causal systems:

$$\hat{g}(z) = (z^2 + 2z - 1)/(z - 0.5)$$

means

$$y[k+1] - 0.5y[k] = u[k+2] + 2u[k+1] - u[k]$$

- State-space model for discrete-time linear systems

$$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{B}[k]\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k]\mathbf{x}[k] + \mathbf{D}[k]\mathbf{u}[k]$$

- State-space model for discrete-time LTI systems

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

$$\begin{aligned} \mathcal{Z}[\mathbf{x}[k+1]] &= \sum_{k=0}^{\infty} \mathbf{x}[k+1] z^{-k} = z \sum_{k=0}^{\infty} \mathbf{x}[k+1] z^{-(k+1)} \\ &= z \left[ \sum_{l=1}^{\infty} \mathbf{x}[l] z^{-l} + \mathbf{x}[0] - \mathbf{x}[0] \right] = z(\hat{\mathbf{x}}(z) - \mathbf{x}[0]) \end{aligned}$$

$$z\hat{\mathbf{x}}(z) - z\mathbf{x}[0] = \mathbf{A}\hat{\mathbf{x}}(z) + \mathbf{B}\hat{\mathbf{u}}(z)$$

$$\hat{\mathbf{y}}(z) = \mathbf{C}\hat{\mathbf{x}}(z) + \mathbf{D}\hat{\mathbf{u}}(z)$$

$$\hat{\mathbf{x}}(z) = (z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{x}[0] + (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\hat{\mathbf{u}}(z)$$

$$\hat{\mathbf{y}}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{x}[0] + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\hat{\mathbf{u}}(z) + \mathbf{D}\hat{\mathbf{u}}(z)$$

$$\boxed{\hat{\mathbf{G}}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}}$$

**Example 2.16 – 1****• Example 2.16:**

Money account w/ interest rate  $r = 0.015\%$  (per day)

$$y[k+1] = y[k] + 0.00015y[k] + u[k+1] = 1.00015y[k] + u[k+1]$$

Define state variable as  $x[k] := y[k]$ , then

$$x[k+1] = 1.00015x[k] + u[k+1]$$

$$y[k] = x[k]$$

not a standard state-space model

but if define  $x[k] := y[k] - u[k]$ , then get

$$y[k+1] = x[k+1] + u[k+1], \quad y[k] = x[k] + u[k]$$

$$x[k+1] = 1.00015x[k] + 1.00015u[k]$$

and

$$y[k] = x[k] + u[k]$$