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# 線性系統 Linear Systems

## Chapter 04 State-Space Solutions & Realizations

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Materials used in these lecture notes are adopted from  
“Linear System Theory & Design,” 3rd. Ed., by C.-T. Chen (1999)

### Outline

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NTUEE-LS4-Solution-2

- Introduction
- Solution of LTI State Equations (4.2)
- Equivalent State Equations (4.3)
- Realizations (4.4)

- Derivative of Exponential Function:

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots$$

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$$

- LTI State Equation and its Solution:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

$$\frac{d}{dt}(e^{-\mathbf{A}t}\mathbf{x}(t)) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

$$e^{-\mathbf{A}\tau}\mathbf{x}(\tau) \Big|_{\tau=0}^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau) d\tau$$

- LTI State Equations:

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{\mathbf{0}}\mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}u(\tau) d\tau$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) = e^{\mathbf{0}}\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}u(\tau) d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}e^{\mathbf{0}}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}u(\tau) d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}u(\tau) d\tau$$

- Useful formulae:

$$\boxed{\frac{\partial}{\partial t} \int_{a(t)}^{b(t)} f(\tau, \tau) d\tau = \int_{a(t)}^{b(t)} \left( \frac{\partial}{\partial t} f(t, \tau) \right) d\tau + \frac{\partial b(t)}{\partial t} f(t, \tau) \Big|_{\tau=b(t)} - \frac{\partial a(t)}{\partial t} f(t, \tau) \Big|_{\tau=a(t)}}$$

$$\boxed{\frac{\partial}{\partial t} \int_{t_0}^t f(t, \tau) d\tau = \int_{t_0}^t \left( \frac{\partial}{\partial t} f(t, \tau) \right) d\tau + f(t, \tau) \Big|_{\tau=t}}$$

- Verification:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}u(\tau) d\tau$$

$$\text{At } t = 0, \quad \mathbf{x}(0) = e^{\mathbf{A}0}\mathbf{x}(0) = e^{\mathbf{0}}\mathbf{x}(0) = \mathbf{I}\mathbf{x}(0) = \mathbf{x}(0)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

$$\dot{\mathbf{x}}(t) = \frac{d}{dt} \left[ e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau \right]$$

$$= \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau + e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) \Big|_{\tau=t}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A} \left( e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau \right) + e^{\mathbf{A}0}\mathbf{B}\mathbf{u}(t)$$

$$= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

• Output Equation:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau + \mathbf{D}\mathbf{u}(t)$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

• Laplace transform:

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)]$$

$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

- How to compute  $e^{At}$
- How to compute  $(sI - A)^{-1}$

### 1. Use Theorem 3.5:

First, compute the eigenvalues of  $A$ ;

Next, find a polynomial  $h(\lambda)$  of deg.  $n - 1$ , s.t.  $h(A) = e^{At}$

2. Use Jordan Form of A:

$$\text{Let } \mathbf{A} = \mathbf{Q} \hat{\mathbf{A}} \mathbf{Q}^{-1}$$

$$\text{Then, } e^{\mathbf{A}t} = \mathbf{Q} e^{\hat{\mathbf{A}}t} \mathbf{Q}^{-1}$$

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

3. Use the infinite power series:

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

4. Use  $e^{\mathbf{At}} = \mathcal{L}^{-1}(s\mathbf{I} - \mathbf{A})^{-1}$

1. Taking the inverse of  $(s\mathbf{I} - \mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})^{-1} &= \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} \\ &= \begin{bmatrix} (s+2)/(s+1)^2 & -1/(s+1)^2 \\ 1/(s+1)^2 & s/(s+1)^2 \end{bmatrix} \end{aligned}$$

## 2. Using Theorem 3.5

eigenvalues of  $\mathbf{A}$  are  $-1, -1$

$$f(\lambda) := (s - \lambda)^{-1} \quad f(-1) = h(-1) : \quad (s + 1)^{-1} = \beta_0 - \beta_1$$

$$h(\lambda) = \beta_0 + \beta_1\lambda \quad f'(-1) = h'(-1) : \quad (s + 1)^{-2} = \beta_1$$

$$\Rightarrow h(\lambda) = [(s + 1)^{-1} + (s + 1)^{-2}] + (s + 1)^{-2}\lambda$$

$$\Rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = h(\mathbf{A}) = [(s + 1)^{-1} + (s + 1)^{-2}]\mathbf{I} + (s + 1)^{-2}\mathbf{A}$$

$$= \begin{bmatrix} (s+2)/(s+1)^2 & -1/(s+1)^2 \\ 1/(s+1)^2 & s/(s+1)^2 \end{bmatrix}$$

## 3. Using $(s\mathbf{I} - \mathbf{A})^{-1} = \mathbf{Q}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\mathbf{Q}^{-1}$ and the Jordan form for $\hat{\mathbf{A}}$

$$\left( s\mathbf{I} - \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \right)^{-1} = \left( \begin{bmatrix} s - \lambda & -1 & 0 \\ 0 & s - \lambda & -1 \\ 0 & 0 & s - \lambda \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} (s - \lambda)^{-1} & (s - \lambda)^{-2} & (s - \lambda)^{-3} \\ 0 & (s - \lambda)^{-1} & (s - \lambda)^{-2} \\ 0 & 0 & (s - \lambda)^{-1} \end{bmatrix}$$

$\Rightarrow$  Every term of  $e^{\hat{\mathbf{A}}t}$  is a linear combination of  
 $e^{\lambda_i t}, \ te^{\lambda_i t}, \ t^2 e^{\lambda_i t}, \ \dots, \ t^{\bar{n}_i - 1} e^{\lambda_i t}, \ i = 1, 2, \dots, m$

**4. Using the infinite power series**

$$(s\mathbf{I} - \mathbf{A})^{-1} = s^{-1}\mathbf{I} + s^{-2}\mathbf{A} + s^{-3}\mathbf{A}^2 + \dots$$

**5. Using the Leverrier algorithm**

(Problem 3.26)

**Example 4.2**

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B} u(\tau) d\tau$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} -\int_0^t (t-\tau) e^{-(t-\tau)} u(\tau) d\tau \\ \int_0^t [1-(t-\tau)] e^{-(t-\tau)} u(\tau) d\tau \end{bmatrix}$$

⇒ Every term of  $e^{\mathbf{A}t} \mathbf{x}(0)$  is a linear combination of  $e^{\lambda_i t}, t e^{\lambda_i t}, t^2 e^{\lambda_i t}, \dots, t^{\bar{n}_i-1} e^{\lambda_i t}$ ,  $i = 1, 2, \dots, m$

- If  $\operatorname{Re}(\lambda_i) < 0$  for all  $i$ ,

then every zero-input response will approach zero as  $t \rightarrow \infty$

- If  $\operatorname{Re}(\lambda_i) > 0$  for some  $i$ ,

then part of zero-input response may grow unbounded as  $t \rightarrow \infty$

$\Rightarrow$  Every term of  $e^{\mathbf{At}}$  is a linear combination of  
 $e^{\lambda_i t}, te^{\lambda_i t}, t^2 e^{\lambda_i t}, \dots, t^{\bar{n}_i - 1} e^{\lambda_i t}, i = 1, 2, \dots, m$

- If  $\operatorname{Re}(\lambda_i) \leq 0$  for all  $i$ , and  $\lambda_j$  with  $\operatorname{Re}(\lambda_j) = 0$  has only index 1,

then zero-input response will be bounded for all  $t$

- If  $\operatorname{Re}(\lambda_i) \leq 0$  for all  $i$ ,

but some  $\lambda_j$  with  $\operatorname{Re}(\lambda_j) = 0$  has index 2 or higher,

then part of zero-input response may grow unbounded as  $t \rightarrow \infty$

$\Rightarrow$  Every term of  $e^{\mathbf{At}}$  is a linear combination of  
 $e^{\lambda_i t}, te^{\lambda_i t}, t^2 e^{\lambda_i t}, \dots, t^{\bar{n}_i - 1} e^{\lambda_i t}, i = 1, 2, \dots, m$

- Finite difference approximation of C.T. systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\dot{\mathbf{x}}(t) = \lim_{T \rightarrow 0} \frac{\mathbf{x}(t+T) - \mathbf{x}(t)}{T} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

- Approximation:

$$\mathbf{x}(t+T) = \mathbf{x}(t) + \mathbf{A}\mathbf{x}(t)T + \mathbf{B}\mathbf{u}(t)T$$

$$\mathbf{x}(kT+T) = (\mathbf{I} + T\mathbf{A})\mathbf{x}(kT) + T\mathbf{B}\mathbf{u}(kT)$$

$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT)$$

- C.T. systems with piecewise constant inputs

$$\mathbf{u}(t) = \mathbf{u}(kT) =: \mathbf{u}[k] \quad \text{for } kT \leq t < (k+1)T$$

(may be generated by computers)

$$\mathbf{u}(t) = \mathbf{u}(kT) =: \mathbf{u}[k] \quad \text{for } kT \leq t < (k+1)T$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

$$\mathbf{x}[k] := \mathbf{x}(kT) = e^{\mathbf{A}kT} \mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

$$\mathbf{x}[k+1] := \mathbf{x}((k+1)T) = e^{\mathbf{A}(k+1)T} \mathbf{x}(0) + \int_0^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

$$\begin{aligned}
 \mathbf{x}[k+1] &:= \mathbf{x}((k+1)T) = e^{\mathbf{A}(k+1)T} \mathbf{x}(0) + \int_0^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau \\
 &= e^{\mathbf{A}T} \left[ e^{\mathbf{A}kT} \mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau \right] \\
 &\quad + \int_{kT}^{(k+1)T} e^{\mathbf{A}(kT+\tau-T)} \mathbf{B} \mathbf{u}(\tau) d\tau \\
 &= e^{\mathbf{A}T} \mathbf{x}[k] + \left( \int_0^T e^{\mathbf{A}\alpha} d\alpha \right) \mathbf{B} \mathbf{u}[k]
 \end{aligned}$$

$$\mathbf{A}_d = e^{\mathbf{A}T} \quad \mathbf{B}_d = \left( \int_0^T e^{\mathbf{A}\tau} d\tau \right) \mathbf{B} \quad \mathbf{C}_d = \mathbf{C} \quad \mathbf{D}_d = \mathbf{D}$$

$$\mathbf{x}[k+1] = \mathbf{A}_d \mathbf{x}[k] + \mathbf{B}_d \mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}_d \mathbf{x}[k] + \mathbf{D}_d \mathbf{u}[k]$$

$$\mathbf{A}_d = e^{\mathbf{A}T} \quad \mathbf{B}_d = \left( \int_0^T e^{\mathbf{A}\tau} d\tau \right) \mathbf{B} \quad \mathbf{C}_d = \mathbf{C} \quad \mathbf{D}_d = \mathbf{D}$$

$$\int_0^T \left( \mathbf{I} + \mathbf{A}\tau + \mathbf{A}^2 \frac{\tau^2}{2!} + \dots \right) d\tau = T\mathbf{I} + \frac{T^2}{2!}\mathbf{A} + \frac{T^3}{3!}\mathbf{A}^2 + \frac{T^4}{4!}\mathbf{A}^3 + \dots$$

- If  $\mathbf{A}$  is **nonsingular**, then

$$\mathbf{A}^{-1} \left( T\mathbf{A} + \frac{T^2}{2!}\mathbf{A}^2 + \frac{T^3}{3!}\mathbf{A}^3 + \dots + \mathbf{I} - \mathbf{I} \right) = \mathbf{A}^{-1}(e^{\mathbf{A}T} - \mathbf{I})$$

$$\boxed{\mathbf{B}_d = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\mathbf{B} \quad (\text{if } \mathbf{A} \text{ is nonsingular})}$$

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

$$e^{\mathbf{M}T} = \mathbf{I} + T\mathbf{M} + \frac{T^2}{2!}\mathbf{M}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} T^k \mathbf{M}^k$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \mathbf{M}^2 = \begin{bmatrix} \mathbf{A}^2 & \mathbf{AB} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \mathbf{M}^3 = \begin{bmatrix} \mathbf{A}^3 & \mathbf{A}^2\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \mathbf{A}_d & \mathbf{B}_d \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \exp \left( \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} T \right)$$

$$\mathbf{A}_d = e^{\mathbf{A}T} \quad \mathbf{B}_d = \left( \int_0^T e^{\mathbf{A}\tau} d\tau \right) \mathbf{B} \quad \mathbf{C}_d = \mathbf{C} \quad \mathbf{D}_d = \mathbf{D}$$

$$\mathbf{x}[k+1] = \mathbf{Ax}[k] + \mathbf{Bu}[k]$$

$$\mathbf{y}[k] = \mathbf{Cx}[k] + \mathbf{Du}[k]$$

$$\mathbf{x}[1] = \mathbf{Ax}[0] + \mathbf{Bu}[0]$$

$$\mathbf{x}[2] = \mathbf{Ax}[1] + \mathbf{Bu}[1] = \mathbf{A}^2\mathbf{x}[0] + \mathbf{ABu}[0] + \mathbf{Bu}[1]$$

⋮

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{Bu}[m]$$

$$\mathbf{y}[k] = \mathbf{CA}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{CA}^{k-1-m}\mathbf{Bu}[m] + \mathbf{Du}[k]$$

$$A = QDQ^{-1} \quad \Rightarrow \quad A^k = QD^kQ^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^2 \end{bmatrix} \quad \begin{bmatrix} \lambda_1^2 & 2\lambda_1 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^2 \end{bmatrix} \quad \begin{bmatrix} \lambda_1^2 & 2\lambda_1 & 1 \\ 0 & \lambda_1^2 & 2\lambda_1 \\ 0 & 0 & \lambda_1^2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_1^k & 0 \\ 0 & 0 & \lambda_1^k \end{bmatrix} \quad \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & 0 \\ 0 & \lambda_1^k & 0 \\ 0 & 0 & \lambda_1^k \end{bmatrix} \quad \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} \\ 0 & 0 & \lambda_1^k \end{bmatrix}$$

- $\lambda_1$ : multiplicity = 4, index = 3
- $\lambda_2$ : multiplicity = 1, index = 1

$$\mathbf{A}^k = \mathbf{Q} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix} \mathbf{Q}^{-1}$$

Every term of  $\mathbf{A}^k \mathbf{x}[0]$  is a linear combination of

$$\lambda_i^k, \quad k\lambda_i^{k-1}, \quad k^2\lambda_i^{k-2}, \dots, \quad k^{\bar{n}_i-1}\lambda_i^{\bar{k}-\bar{n}_i+1}, \quad i = 1, 2, \dots, m.$$

$$\mathbf{A}^k = \mathbf{Q} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix} \mathbf{Q}^{-1}$$

- If  $|\lambda_i| < 1$  for all  $i$ , then every zero-input response will approach zero as  $k \rightarrow \infty$
- If  $|\lambda_i| > 1$  for some  $i$ , then part of zero-input response may grow unbounded as  $k \rightarrow \infty$
- If  $|\lambda_i| \leq 1$  for all  $i$ , and  $\lambda_j$  with  $|\lambda_j| = 1$  has only index 1, then zero-input response will be bounded for all  $k$ .
- If  $|\lambda_i| \leq 1$  for all  $i$ , but some  $\lambda_j$  with  $|\lambda_j| = 1$  has index 2 or higher, then part of zero-input response may grow unbounded as  $k \rightarrow \infty$ .

## In Summary: CT

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

$$\dot{x}(t) = ax(t) + bu(t) \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau$$

$$s\hat{x}(s) - x(0) = a\hat{x}(s) + b\hat{u}(s)$$

$$\hat{x}(s) = \frac{1}{s-a}x(0) + \frac{b}{s-a}\hat{u}(s)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

$$s\hat{\mathbf{x}}(s) - \mathbf{x}(0) = \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s)$$

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s)$$

## In Summary: DT

$$\begin{cases} \mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k] \end{cases}$$

$$x[k+1] = ax[k] + bu[k]$$

$$x[k] = a^k x[0] + \sum_{m=0}^{k-1} a^{k-1-m} bu[m]$$

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{x}[k] = \mathbf{A}^k \mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m} \mathbf{B}\mathbf{u}[m]$$

## In Summary: A in CT & DT

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$A = QDQ^{-1} \Rightarrow e^{At} = Qe^{Dt}Q^{-1}$$

$$D = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & (t^2/2!)e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$\begin{cases} \mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k] \end{cases}$$

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]$$

$$A = QDQ^{-1} \Rightarrow A^k = QD^kQ^{-1}$$

$$D = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad D^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & (k(k-1)/2)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

## In Summary: exp(A)

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

$$e^{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} e^0 & e^0 \\ e^0 & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} ?$$

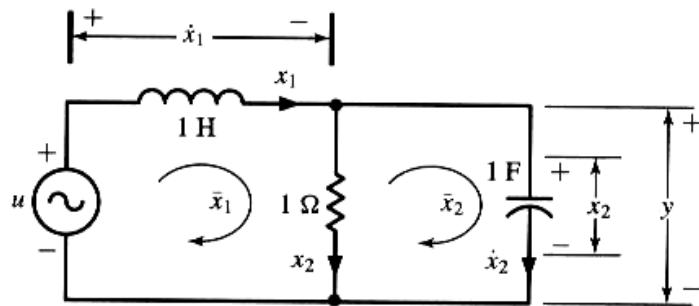
$$e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} ?$$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} ?$$

$$e^{\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix}} = \begin{bmatrix} e^{\mathbf{A}} & 0 \\ 0 & e^{\mathbf{B}} \end{bmatrix} ?$$

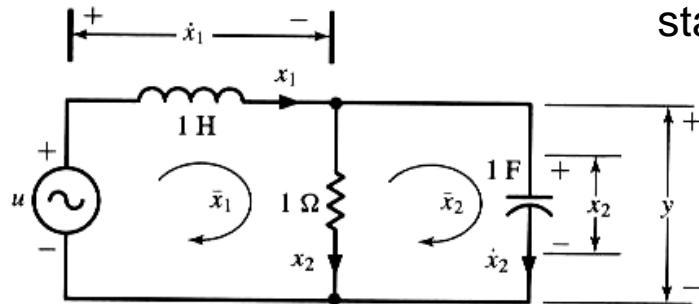
$$\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix}^k = \begin{bmatrix} \mathbf{A}^k & 0 \\ 0 & \mathbf{B}^k \end{bmatrix} ?$$

- Example 4.3: Equivalent state equations



- Example 4.3: Equivalent state equations

Two sets of state variables:



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \ 1] \mathbf{x}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [1 \ -1] \bar{\mathbf{x}}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{D} \mathbf{u}(t) \end{cases} \quad \begin{cases} \dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}} \bar{\mathbf{x}}(t) + \bar{\mathbf{B}} \mathbf{u}(t) \\ \mathbf{y}(t) = \bar{\mathbf{C}} \bar{\mathbf{x}}(t) + \bar{\mathbf{D}} \mathbf{u}(t) \end{cases}$$

**Definition 4.1** Let  $\mathbf{P}$  be an  $n \times n$  real nonsingular matrix and let  $\bar{\mathbf{x}} = \mathbf{Px}$ . Then the state equation,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{Ax}(t) + \mathbf{Bu}(t) & \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{Cx}(t) + \mathbf{Du}(t) & \text{and} & \\ & & \bar{\mathbf{y}}(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\mathbf{u}(t) \end{aligned}$$

where

$$\bar{\mathbf{A}} = \mathbf{PAP}^{-1} \quad \bar{\mathbf{B}} = \mathbf{PB} \quad \bar{\mathbf{C}} = \mathbf{CP}^{-1} \quad \bar{\mathbf{D}} = \mathbf{D}$$

are said to be (algebraically) equivalent, and

$\bar{\mathbf{x}} = \mathbf{Px}$  is called an equivalence transformation.

Define  $\mathbf{P}^{-1} = \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n]$

$$\mathbf{AQ} = \mathbf{Q}\bar{\mathbf{A}} \iff \text{i-th column of } \bar{\mathbf{A}}, \text{ i.e., } \bar{\mathbf{a}}_i,$$

From Sec 3.4

is the representation of  $\mathbf{Aq}_i$  w.r.t  $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$

$$\mathbf{B} = \mathbf{Q}\bar{\mathbf{B}} \iff \text{i-th column of } \bar{\mathbf{B}}, \text{ i.e., } \bar{\mathbf{b}}_i$$

is the representation of  $\mathbf{b}_i$  w.r.t  $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$

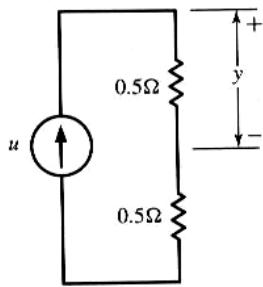
$$\begin{bmatrix} \mathbf{b}_{1i} \\ \mathbf{b}_{2i} \\ \vdots \\ \mathbf{b}_{ni} \end{bmatrix} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n] \begin{bmatrix} \bar{\mathbf{b}}_{1i} \\ \bar{\mathbf{b}}_{2i} \\ \vdots \\ \bar{\mathbf{b}}_{ni} \end{bmatrix}$$

- Equivalent state equations

have the same eigenvalues and transfer matrix:

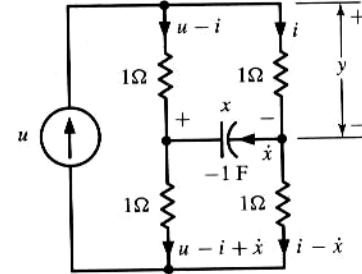
- Two state equations may have the same transfer matrix  
(and are called *zero-state equivalent*),  
but are NOT algebraically equivalent.

### Example 4.4



$$y(t) = 0.5 \cdot u(t)$$

$$\hat{y}(s) = 0.5\hat{u}(s)$$



$$\dot{x}(t) = x(t)$$

$$y(t) = 0.5x(t) + 0.5u(t)$$

$$\hat{y}(s) = 0.5\hat{u}(s)$$

### Theorem 4.1

$$(s\mathbf{I} - \mathbf{A})^{-1} = s^{-1}\mathbf{I} + s^{-2}\mathbf{A} + s^{-3}\mathbf{A}^2 + \dots$$

$$\mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \bar{\mathbf{D}} + \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}$$

$$\Leftrightarrow \mathbf{D} + \mathbf{C}\mathbf{B}s^{-1} + \mathbf{C}\mathbf{A}\mathbf{B}s^{-2} + \mathbf{C}\mathbf{A}^2\mathbf{B}s^{-3} + \dots$$

$$= \bar{\mathbf{D}} + \bar{\mathbf{C}}\bar{\mathbf{B}}s^{-1} + \bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{B}}s^{-2} + \bar{\mathbf{C}}\bar{\mathbf{A}}^2\bar{\mathbf{B}}s^{-3} + \dots$$

### Theorem 4.1

Two linear time-invariant state equations  $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$  and  $\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}\}$  are zero-state equivalent or have the same transfer matrix if and only if  $\mathbf{D} = \bar{\mathbf{D}}$  and

$$\mathbf{C}\mathbf{A}^m\mathbf{B} = \bar{\mathbf{C}}\bar{\mathbf{A}}^m\bar{\mathbf{B}} \quad m = 0, 1, 2, \dots$$

## Diagonal/Jordan Canonical Form (4.3.1) – 1

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{44} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\mathbf{Q}_1^{-1} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{44} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \mathbf{Q}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a + jb & 0 \\ 0 & 0 & 0 & a - jb \end{bmatrix}$$

$$\mathbf{Q}_1^{-1} \mathbf{A} \mathbf{Q}_1 = \mathbf{A}_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & j & -j \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a + jb & 0 \\ 0 & 0 & 0 & a - jb \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix}$$

$$\mathbf{Q}_2^{-1} \mathbf{A}_1 \mathbf{Q}_2 = \mathbf{A}_2 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix}$$

## Diagonal/Jordan Canonical Form – 2

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{44} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a + jb & 0 \\ 0 & 0 & 0 & a - jb \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{P} \mathbf{A} \mathbf{P}^{-1}$$

$$\mathbf{A}_2 = \mathbf{Q}_2^{-1} \mathbf{Q}_1^{-1} \mathbf{A} \mathbf{Q}_1 \mathbf{Q}_2 = \bar{\mathbf{A}}$$

$$\mathbf{P}^{-1} = \mathbf{Q}_1 \mathbf{Q}_2 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \operatorname{Re}(\mathbf{q}_3) & \operatorname{Im}(\mathbf{q}_3) \end{bmatrix}$$

$\mathbf{Q} = [ \text{n L.I. eigenvectors/generalized eigenvectors} ]$

$\Rightarrow \bar{\mathbf{A}}$  has the diagonal/Jordan form, which may have complex elements.

$$\mathbf{J} := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$$

For real  $\bar{\mathbf{A}}$ , the modal form may be obtained with a further equivalence transformation:

$$\begin{aligned} \bar{\mathbf{Q}}^{-1} \mathbf{J} \bar{\mathbf{Q}} &:= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & j & -j \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} \\ &\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix} =: \bar{\mathbf{A}} \end{aligned}$$

- Combined equivalence transformation for the modal form:

$$\begin{aligned} \mathbf{P}^{-1} &= \mathbf{Q} \bar{\mathbf{Q}} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3 \ \mathbf{q}_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix} \\ &= [\mathbf{q}_1 \ \mathbf{q}_2 \ \operatorname{Re}(\mathbf{q}_3) \ \operatorname{Im}(\mathbf{q}_3)] \end{aligned}$$

- Please figure out the modal form and the associated equivalence transformation for systems with generalized eigenvectors of grades larger than unity

- Special algebraically equivalent forms –

the canonical form & companion canonical form:

If  $Q := [ b_1 \ Ab_1 \ \cdots \ A^{n-1}b_1 ]$  is nonsingular

Then  $\bar{A} = Q^{-1}AQ$  has the **companion** form

And  $\bar{b}_1 = Q^{-1}b_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

- Magnitude scaling:

in **hardware** (op-amp circuit)

or **software** (digital computation) simulations,

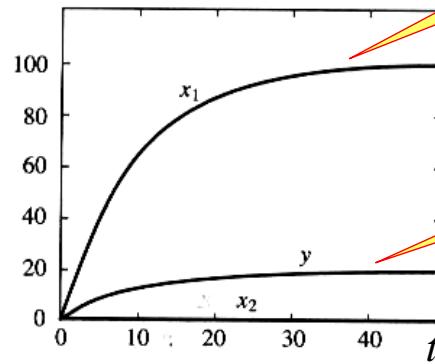
usually need to ensure that

**magnitude** of all signals are **not too large and not too small**

### Example 4.5

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.1 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0.1 \end{bmatrix} u$$
$$y = [0.1 \quad -1] \mathbf{x}$$

unit-step



$|x_1|$  too large

$|x_2|$  too small

### Example 4.5

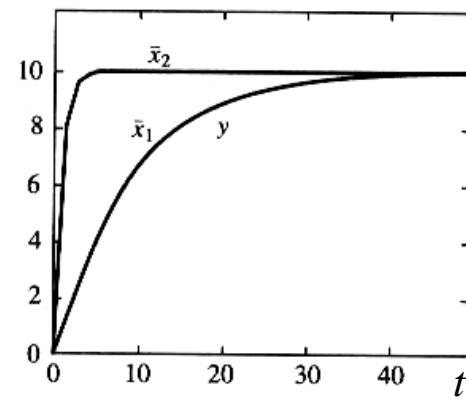
Use state variable change to adjust the magnitudes:

$$\bar{x}_1 = \frac{20}{100}x_1 = 0.2x_1 \quad \bar{x}_2 = \frac{20}{0.1}x_2 = 200x_2$$

$$\bar{\mathbf{x}} = \mathbf{Px} \quad \mathbf{P} = \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix}$$

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -0.1 & 0.002 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 20 \end{bmatrix} u$$

$$y = [1 \quad -0.005]\bar{\mathbf{x}}$$



- A transfer matrix  $\hat{\mathbf{G}}(s)$  is **realizable**  
if there exists a **finite dimensional** state equation  $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ ,  
a **realization** of  $\hat{\mathbf{G}}(s)$ , such that  $\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

### Special Case: Single-Input-Single-Output Systems

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

$$\hat{\mathbf{y}} = \frac{n(s)}{d(s)} \hat{\mathbf{u}}$$

$$\hat{\mathbf{y}} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \hat{\mathbf{u}}$$

$$\hat{\mathbf{x}}_1 = \frac{s^3}{d(s)} \hat{\mathbf{u}} \quad \hat{\mathbf{y}} = b_1 \hat{\mathbf{x}}_1 + b_2 \hat{\mathbf{x}}_2 + b_3 \hat{\mathbf{x}}_3 + b_4 \hat{\mathbf{x}}_4$$

$$\hat{\mathbf{x}}_2 = \frac{s^2}{d(s)} \hat{\mathbf{u}} \quad s\hat{\mathbf{x}}_4 = \frac{s}{d(s)} \hat{\mathbf{u}} = \hat{\mathbf{x}}_3$$

$$\hat{\mathbf{x}}_3 = \frac{s}{d(s)} \hat{\mathbf{u}} \quad s\hat{\mathbf{x}}_3 = \frac{s^2}{d(s)} \hat{\mathbf{u}} = \hat{\mathbf{x}}_2$$

$$\hat{\mathbf{x}}_4 = \frac{1}{d(s)} \hat{\mathbf{u}} \quad s\hat{\mathbf{x}}_2 = \frac{s^3}{d(s)} \hat{\mathbf{u}} = \hat{\mathbf{x}}_1$$

$$s^4 \hat{x}_4 + a_1 s^3 \hat{x}_4 + a_2 s^2 \hat{x}_4 + a_3 s \hat{x}_4 + a_4 \hat{x}_4 = \hat{u}$$

$$s \hat{x}_1 + a_1 \hat{x}_1 + a_2 \hat{x}_2 + a_3 \hat{x}_3 + a_4 \hat{x}_4 = \hat{u}$$

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 - a_3 x_3 - a_4 x_4 + u$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = x_2$$

$$\dot{x}_4 = x_3$$

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [b_1 \ b_2 \ b_3 \ b_4] x$$

Special Case: Single-Input-Multi-Output Systems ( $p = 1$ )

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \frac{1}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \begin{bmatrix} \beta_{11}s^3 + \beta_{12}s^2 + \beta_{13}s + \beta_{14} \\ \beta_{21}s^3 + \beta_{22}s^2 + \beta_{23}s + \beta_{24} \end{bmatrix}$$

↓ a realization

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix} x + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} u$$

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{4s - 10}{2s + 1} & \frac{3}{s + 2} \\ \frac{1}{(2s + 1)(s + 2)} & \frac{s + 2}{s + 1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-12}{2s + 1} & \frac{3}{s + 2} \\ \frac{1}{(2s + 1)(s + 2)} & \frac{s + 2}{(s + 2)^2} \end{bmatrix}$$

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{s^3 + 4.5s^2 + 6s + 2} \begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix}$$

$$= \frac{1}{d(s)} \left( \underbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}_{\mathbf{N}_1} s^2 + \underbrace{\begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}}_{\mathbf{N}_2} s + \underbrace{\begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}}_{\mathbf{N}_3} \right)$$

$\overbrace{\quad\quad\quad}^{\mathbf{N}(s)}$

$$\dot{\mathbf{x}} = \begin{bmatrix} -4.5 & 0 & \vdots & -6 & 0 & \vdots & -2 & 0 \\ 0 & -\alpha_1 \mathbf{I}_2 & -4.5 & \vdots & 0 & -\alpha_2 \mathbf{I}_2 & -6 & \vdots & 0 & -\alpha_3 \mathbf{I}_2 & -2 \\ \dots & \dots \\ 1 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots \\ 0 & 0 & \vdots & 1 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 1 & \vdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dots & \dots \\ 0 & 0 \\ 0 & 0 \\ \dots & \dots \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -6 & 3 & \vdots & -24 & 7.5 & \vdots & -24 & 3 \\ 0 & \mathbf{N}_1 & 1 & \vdots & 0.5 & 1.5 & \vdots & 1 & 0.5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

a six-dimensional realization

**Theorem 4.2**

A transfer matrix  $\hat{\mathbf{G}}(s)$  is realizable if and only if  $\hat{\mathbf{G}}(s)$  is a proper rational matrix.

**Proof:**

“ $\Rightarrow$ ”

$$\hat{\mathbf{G}}_{sp}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{\det(s\mathbf{I} - \mathbf{A})} \mathbf{C}[\text{Adj } (s\mathbf{I} - \mathbf{A})]\mathbf{B} : \text{strictly proper}$$

$$\rightarrow \hat{\mathbf{G}}(\infty) = \mathbf{D} \quad \text{and} \quad \hat{\mathbf{G}}(s) \text{ is proper}$$

“ $\Leftarrow$ ” Let  $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s)$

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r$$

: monic l.c.d. of all entries of  $\hat{\mathbf{G}}_{sp}(s)$

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)} [\mathbf{N}(s)] = \frac{1}{d(s)} [\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1} s + \mathbf{N}_r]$$

- Let us check the transfer matrix of the following state equation in controllable canonical form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 \mathbf{I}_p & -\alpha_2 \mathbf{I}_p & \cdots & -\alpha_{r-1} \mathbf{I}_p & -\alpha_r \mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}_{rp \times rp} \mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}_{rp \times p} \mathbf{u}$$

**A**

$$\mathbf{y} = [\mathbf{N}_1 \ \mathbf{N}_2 \ \cdots \mathbf{N}_{r-1} \ \mathbf{N}_r]_{q \times rp} \mathbf{x} + \hat{\mathbf{G}}(\infty) \mathbf{u}$$

**C**

**B**

- Consider

$$\mathbf{Z} := \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_r \end{bmatrix} := (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}$$

i.e.,  $s\mathbf{Z} = \mathbf{AZ} + \mathbf{B}$

→  $s\mathbf{Z}_2 = \mathbf{Z}_1, \quad s\mathbf{Z}_3 = \mathbf{Z}_2, \quad \dots, \quad s\mathbf{Z}_r = \mathbf{Z}_{r-1}$

i.e.,  $\mathbf{Z}_2 = \frac{1}{s}\mathbf{Z}_1, \quad \mathbf{Z}_3 = \frac{1}{s^2}\mathbf{Z}_1, \quad \dots, \quad \mathbf{Z}_r = \frac{1}{s^{r-1}}\mathbf{Z}_1$

Also

$$\begin{aligned}s\mathbf{Z}_1 &= -\alpha_1 \mathbf{Z}_1 - \alpha_2 \mathbf{Z}_2 - \cdots - \alpha_r \mathbf{Z}_r + \mathbf{I}_p \\ &= -\left(\alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right) \mathbf{Z}_1 + \mathbf{I}_p\end{aligned}$$

i.e.,

$$\mathbf{Z}_1 = \frac{s^{r-1}}{d(s)} \mathbf{I}_p, \quad \mathbf{Z}_2 = \frac{s^{r-2}}{d(s)} \mathbf{I}_p, \quad \dots, \quad \mathbf{Z}_r = \frac{1}{d(s)} \mathbf{I}_p$$

→  $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \hat{\mathbf{G}}(\infty) = \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_r] + \hat{\mathbf{G}}(\infty)$

i.e.,  $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \hat{\mathbf{G}}(\infty)\}$  is a **realization** of  $\hat{\mathbf{G}}(s)$

In addition to the controllable canonical form,  
there is also the “**observable**” canonical form. See Prob. 4.9

### Special Case: Single-Input Systems ( $p = 1$ )

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \frac{1}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \begin{bmatrix} \beta_{11} s^3 + \beta_{12} s^2 + \beta_{13} s + \beta_{14} \\ \beta_{21} s^3 + \beta_{22} s^2 + \beta_{23} s + \beta_{24} \end{bmatrix}$$

↓ a realization

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix} \mathbf{x} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} u$$

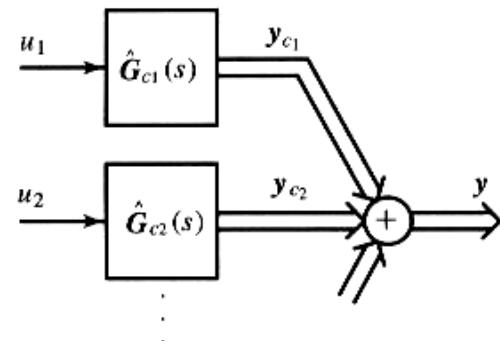
A multi-input LTI system is the sum of many single-input LTI systems, so can realize each single-input subsystem and form the sum:

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s) = \hat{\mathbf{G}}_{c1}(s)\hat{u}_1(s) + \hat{\mathbf{G}}_{c2}(s)\hat{u}_2(s) + \dots =: \hat{\mathbf{y}}_{c1}(s) + \hat{\mathbf{y}}_{c2}(s) + \dots$$

1st and 2nd columns of  $\hat{\mathbf{G}}(s)$

## Example 4.7

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{4s - 10}{2s + 1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$



$$\dot{\mathbf{x}}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$\mathbf{y}_{c1} = \mathbf{C}_1 \mathbf{x}_1 + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2 \mathbf{x}_2 + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\mathbf{y}_{c2} = \mathbf{C}_2 \mathbf{x}_2 + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2$$

### Example 4.7

$$\dot{\mathbf{x}}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$\mathbf{y}_{c1} = \mathbf{C}_1 \mathbf{x}_1 + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2 \mathbf{x}_2 + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\mathbf{y}_{c2} = \mathbf{C}_2 \mathbf{x}_2 + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2$$

Overall Realization:

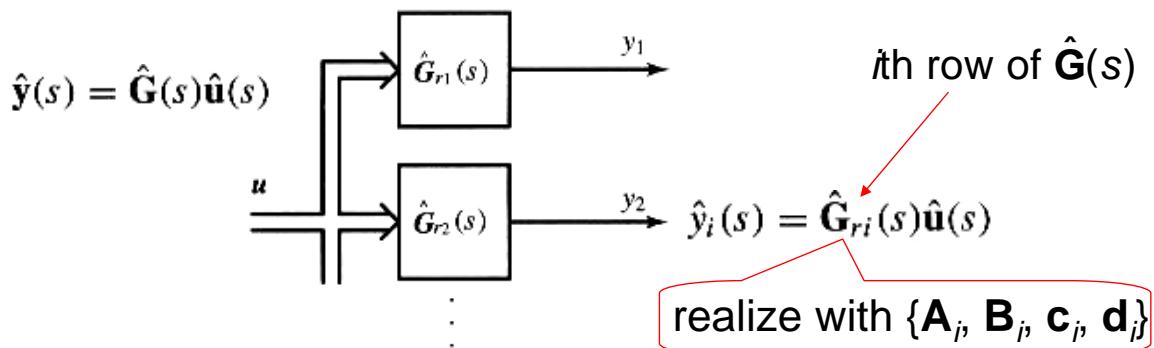
$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \mathbf{y}_{c1} + \mathbf{y}_{c2} = [\mathbf{C}_1 \ \mathbf{C}_2] \mathbf{x} + [\mathbf{d}_1 \ \mathbf{d}_2] \mathbf{u}$$

For this case, a four-dimensional realization with this method

### Example 4.7

- Can also focus on the realizations of single-output systems, then treat LTI systems with multi-outputs as combinations of single-outputs subsystems.



Overall Realization:

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_q \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_q \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{c}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{c}_q \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_q \end{bmatrix} \mathbf{u}$$

- Realizations for **discrete-time** systems:  
all discussions apply,  
except “**s**” is changed to “**z**”, “**x(t)**” is changed to “**x(k)**”,  
and “**dx(t)/dt**” is changed to “**x(k+1)**”.