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線性系統
Linear Systems

Chapter 05
Stability

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Materials used in these lecture notes are adopted from
"Linear System Theory & Design," 3rd. Ed., by C.-T. Chen (1999)

Outline

- Introduction
- Input-Output Stability (5.2)
- Internal Stability (5.3)
- Lyapunov Theorem (5.4)

Basic Concept (5.1)

- Definition:

$$in \longrightarrow \boxed{\text{System}} \longrightarrow out$$

- Stable Systems
  - ➡ Bounded Input and Bounded Output (BIBO) Stability
- Unstable Systems
  - ➡ Small Input generates Unbounded Output
- Response of Linear Systems
  - ➡ Zero-State Response + Zero-Input Response
  - ➡ BIBO Stability + Marginal/Asymptotic Stability
  - ➡ Input-Output Stability + Internal Stability

Input-Output Stability (5.2)

- Definition:
- Bounded Signal s(t):
  - ➡ $\exists$ a constant bound $b < \infty$, s.t. $|s(t)| \leq b$, $\forall t \geq 0$
- Bounded-Input-Bounded-Output (BIBO) Stable Systems:
  - ➡ EVERY bounded input excites a bounded output
  - ➡ Bounded outputs in response to ALL bounded inputs
    - ➡ (zero-state response only)

$$in \longrightarrow \boxed{\text{System}} \longrightarrow out$$

- An SISO causal LTI systems:

$$y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau = \int_0^t g(\tau)\, u(t-\tau)\, d\tau$$

Theorem 5.1 unit-impulse response

A SISO system described by $g(t)$ is BIBO stable if and only if $g(t)$ is absolutely integrable in $[0, \infty)$, or

$$\int_0^\infty |g(t)|dt \le M < \infty$$

for some constant M.

Proof: "$\Leftarrow$" $y(t) = \int_0^t g(\tau) \, u(t-\tau) \, d\tau$

$$\int_0^t g(\tau) \, u(t-\tau) \, d\tau$$

$$\int_0^t g(\tau) \, d\tau$$

Proof: "$\Rightarrow$"

Suppose $\int_0^{t_1} |g(\tau)| \, d\tau = \infty$

Then for the bounded input $u(t)$ with

$$u(t_1 - \tau) = \begin{cases} 1 & \text{if } g(\tau) \ge 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases}$$

The corresponding output is unbounded at $t = t_1$, as

$$y(t_1) = \int_0^{t_1} g(\tau) u(t_1 - \tau) \, d\tau = \int_0^{t_1} |g(\tau)| \, d\tau = \infty \qquad \boxtimes$$

(Contradiction)

Theorem 5.1 – 3

- An absolutely integrable function may not approach zero!
  - That is,

$$\int_0^\infty |f(t)|\,dt < \infty \quad \overset{?}{\Rightarrow} \quad \lim_{t\to\infty} f(t) = 0$$

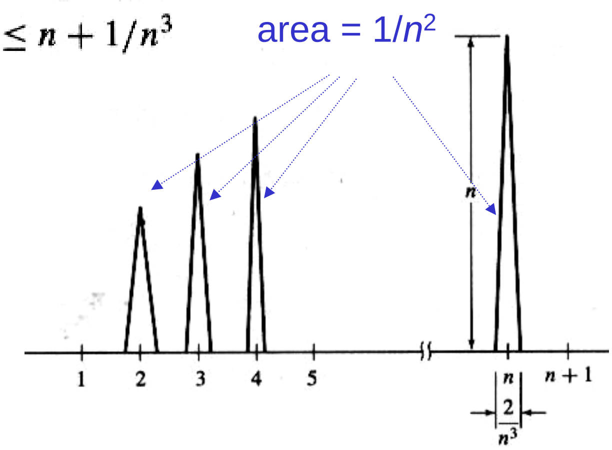
- For example,

$$f(t-n) = \begin{cases} n + (t-n)n^4 & \text{for } n - 1/n^3 \le t \le n \\ n - (t-n)n^4 & \text{for } n < t \le n + 1/n^3 \end{cases}$$

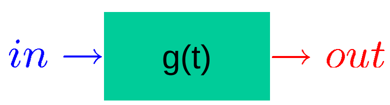
$$n = 2, 3, \ldots$$

$$\int_0^\infty |f(t)|\,dt = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty,$$

but $\lim_{t\to\infty} f(t) \neq 0$



Theorem 5.2 (5.2)



Theorem 5.2

If a system with impulse response $g(t)$ is BIBO stable, then, as $t \to \infty$:

1. The output excited by $u(t) = a$, for $t \ge 0$, approaches $\hat{g}(0) \cdot a$.
2. The output excited by $u(t) = \sin \omega_o t$, for $t \ge 0$, approaches

$$|\hat{g}(j\omega_o)| \sin(\omega_o t + \measuredangle \hat{g}(j\omega_o))$$

where $\hat{g}(s)$ is the Laplace transform of $g(t)$ or

$$\hat{g}(s) = \int_0^\infty g(\tau) e^{-s\tau}\, d\tau$$

Proof:

1. $$y(t) = \int_0^t g(\tau)\ u(t-\tau)\ d\tau$$

Proof:

$$\hat{g}(s) = \int_0^\infty g(\tau)\, e^{-s\tau}\, d\tau$$

$$\hat{g}(jw) = \int_0^\infty g(\tau) \left[\cos w\tau - j \sin w\tau\right] d\tau$$

$$= \int_0^\infty g(\tau) \cos w\tau + j\ g(\tau)\ (-\sin w\tau)\ d\tau$$

2. $$y(t) = \int_0^t g(\tau)\ \ \sin w_0(t-\tau)\ \ d\tau$$

$$= \int_0^t g(\tau)$$

Theorem 5.2 – 4

as $t \longrightarrow \infty$

$$y(t) \longrightarrow \sin w_0 t \int_0^\infty g(\tau)\cos w_0\tau d\tau - \cos w_0 t \int_0^\infty g(\tau)\sin w_0\tau d\tau$$

Theorem 5.3 (5.2)

Theorem 5.3

A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part or, equivalently, lies inside the left-half s-plane.

Proof:

Irrational Transfer Function (5.2)

- Example 5.1: (Fig. 2.5(a))

$$g(t) = \sum_{i=1}^{\infty} a^i \delta(t-i) \qquad \hat{g}(s) = \frac{ae^{-s}}{1-ae^{-s}} \qquad \text{: irrational}$$

$$\int_0^{\infty} |g(t)|\,dt = \sum_{i=1}^{\infty} |a|^i = \begin{cases} \infty & \text{if } |a| \geq 1 \\ |a|/(1-|a|) < \infty & \text{if } |a| < 1 \end{cases}$$

Input-Output Stability: MIMO Case (5.2)

Theorem 5.M1

A multivariable system with impulse response matrix $\mathbf{G}(t) = [g_{ij}(t)]$ is BIBO stable if and only if every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.

Theorem 5.M3

A multivariable system with proper rational transfer matrix $\hat{\mathbf{G}}(s) = [\hat{g}_{ij}(s)]$ is BIBO stable if and only if every pole of every $\hat{g}_{ij}(s)$ has a negative real part.

- State Equations and Transfer Function:

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

$$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- Poles and Eigenvalues:

$$\hat{\mathbf{G}}(s) = \frac{1}{\det(s\mathbf{I} - \mathbf{A})} \mathbf{C}\left[\mathrm{Adj}(s\mathbf{I} - \mathbf{A})\right]\mathbf{B} + \mathbf{D}$$

Therefore, $\left\{ \text{poles of } \hat{\mathbf{G}}(s) \right\}$ $\left\{ \text{eigenvaluses of } \mathbf{A} \right\}$

- Example 5.2: (from Example 4.4, Fig. 4.2(b) on p. 96)

$$\dot{x}(t) = x(t) + 0 \cdot u(t) \qquad y(t) = 0.5x(t) + 0.5u(t)$$

➡ One eigenvalue: +1

$$\hat{g}(s) = 0.5(s-1)^{-1} \cdot 0 + 0.5 = 0.5$$

➡ No poles, BIBO stable

➡ BIBO stable ---(?)--- zero-input response

- A DT SISO system:

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m]$$

- Boundedness of signal s[k]:
  - ➡ ∃ a bound $b < \infty$, s.t. $|s[k]| \le b$, $\forall k \ge 0$

- Bounded-Input-Bounded-Output (BIBO) Systems:
  - ➡ Every bounded input excites a bounded output
  - ➡ Bounded outputs in response to all bounded inputs
    - ➡ (zero-state response only)

$$in \longrightarrow \boxed{\text{System}} \longrightarrow out$$

Theorem 5.D1

A discrete-time SISO system described by $g[k]$ is BIBO stable if and only if $g[k]$ is absolutely summable in $[0, \infty)$ or

$$\sum_{k=0}^{\infty} |g[k]| \le M < \infty$$

for some constant M.

Theorem 5.D2

If a discrete-time system with impulse response sequence $g[k]$ is BIBO stable, then, as $k \to \infty$:

1. The output excited by $u[k] = a$, for $k \ge 0$, approaches $\hat{g}(1) \cdot a$.
2. The output excited by $u[k] = \sin \omega_o k$, for $k \ge 0$, approaches

$$|\hat{g}(e^{j\omega_o})| \sin(\omega_o k + \measuredangle \hat{g}(e^{j\omega_o}))$$

where $\hat{g}(z)$ is the z-transform of $g[k]$ or

$$\hat{g}(z) = \sum_{m=0}^{\infty} g[m] z^{-m}$$

Proof:

d.c. gain

1. $\lim_{k\to\infty} y(t) = \lim_{k\to\infty} \sum_0^k g(m)u(k-m) = \sum_0^\infty g(m)a = a\sum_0^\infty g(m) = a\boxed{\hat{g}(1)}$

2.

$$y[k] = \sum_{m=0}^{k} g[m]\sin\omega_o[k-m]$$
$$= \sum_{m=0}^{k} g[m](\sin\omega_o k\cos\omega_o m - \cos\omega_o k\sin\omega_o m)$$
$$= \sin\omega_o k\sum_{m=0}^{k} g[m]\cos\omega_o m - \cos\omega_o k\sum_{m=0}^{k} g[m]\sin\omega_o m$$

$$\text{as } k\to\infty,\quad y[k]\to \sin\omega_o k\underbrace{\sum_{m=0}^{\infty} g[m]\cos\omega_o m}_{\mathrm{Re}\{\hat{g}(e^{j\omega_o})\} = |\hat{g}(e^{j\omega_o})|\cos[\angle\hat{g}(e^{j\omega_o})]} - \cos\omega_o k\underbrace{\sum_{m=0}^{\infty} g[m]\sin\omega_o m}_{-\mathrm{Im}\{\hat{g}(e^{j\omega_o})\} = -|\hat{g}(e^{j\omega_o})|\sin[\angle\hat{g}(e^{j\omega_o})]}$$

Theorem 5.D3

A discrete-time SISO system with proper rational transfer function $\hat{g}(z)$ is BIBO stable if and only if every pole of $\hat{g}(z)$ has a magnitude less than 1 or, equivalently, lies inside the unit circle on the z-plane.

Proof:

$$\hat{g}(z) = \sum_i\left[\frac{r_{i1}}{z-p_i} + \frac{r_{i2}}{(z-p_i)^2} + \cdots + \frac{r_{im_i}}{(z-p_i)^{m_i}}\right]$$
$$\Rightarrow g[k] = \sum_i\left(r_{i1} + r_{i2}k + \cdots + r_{im_i}k^{m_i-1}\right)p_i^k$$
$$\therefore \sum_0^\infty |g[k]| < \infty \quad\Leftrightarrow\quad |p_i| < 1 \qquad \forall i$$

Theorem 5.MD1

A MIMO discrete-time system with impulse response sequence matrix $\mathbf{G}[k] = [g_{ij}[k]]$ is BIBO stable if and only if every $g_{ij}[k]$ is absolutely summable.

Theorem 5.MD3

A MIMO discrete-time system with discrete proper rational transfer matrix $\hat{\mathbf{G}}(z) = [\hat{g}_{ij}(z)]$ is BIBO stable if and only if every pole of every $\hat{g}_{ij}(z)$ has a magnitude less than 1.

- State Equations and Transfer Function:

$$\hat{\mathbf{y}}(z) = \hat{\mathbf{G}}(z)\hat{\mathbf{u}}(z)$$

$$\hat{\mathbf{G}}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\begin{cases} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k] \end{cases}$$

- Poles of a discrete transfer matrix
  form a subset of the eigenvalues of any of its realizations:

$$\hat{\mathbf{G}}(z) = \frac{1}{\det(z\mathbf{I} - \mathbf{A})}\mathbf{C}[\mathrm{Adj}(z\mathbf{I} - \mathbf{A})]\mathbf{B} + \mathbf{D}$$

- Boundedness/Convergence of zero-input response x(t):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{u}(t) \equiv 0, \quad \mathbf{x}(0) = \mathbf{x}_0$$

$$\Rightarrow \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \Leftrightarrow \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Definition 5.1 *The zero-input response of (5.4) or the equation* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is* marginally stable *or* stable in the sense of Lyapunov *if every finite initial state* $\mathbf{x}_o$ *excites a bounded response. It is* asymptotically stable *if every finite initial state excites a bounded response, which, in addition, approaches* $\mathbf{0}$ *as* $t \to \infty$.

Theorem 5.4

1. The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of $\mathbf{A}$.

2. The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have negative real parts.

Proof: algebraic equivalence transformation preserves stability

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases} \qquad \begin{cases} \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\,\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \bar{\mathbf{C}}\,\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\,\mathbf{u}(t) \end{cases}$$

$\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ or $\mathbf{x} = \mathbf{P}^{-1}\bar{\mathbf{x}}$ **P**: nonsingular

$\therefore \;\; \|\bar{\mathbf{x}}\| \le \|\mathbf{P}\|\|\mathbf{x}\|$ and $\|\mathbf{x}\| \le \|\mathbf{P}^{-1}\|\|\bar{\mathbf{x}}\|$

$\therefore \;\; \|\bar{\mathbf{x}}(t)\| < \infty, \;\; \forall t \quad \Leftrightarrow \quad \|\mathbf{x}(t)\| < \infty, \;\; \forall t$

And $\lim_{t\to 0} \bar{\mathbf{x}}(t) = \mathbf{0} \quad \Leftrightarrow \quad \lim_{t\to 0} \mathbf{x}(t) = \mathbf{0}$

$\therefore$ Consider $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{x}}$, where $\bar{\mathbf{A}}$ is in Jordan form,

And $e^{\bar{\mathbf{A}}t}$ is a block diagonal matrix with diagonal blocks like

$$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2e^{\lambda_1 t}/2! & t^3e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

$\therefore$ Part **2** of the Theorem is obvious (all exponential terms decay)

For Part **1**, the conditions imply

either the exponential term decays to 0 (Re(e-value) < 0),

or exponential term is bounded (Re(e-value) = 0) and

the corresponding Jordan blocks are all 1×1

Example 5.4 (5.3)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}$$

$$\begin{cases} \Delta(\lambda) &= \lambda^2(\lambda+1) \\ \Phi(\lambda) &= \lambda(\lambda+1) \end{cases}$$

➡ Marginally Stable

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}$$

$$\begin{cases} \Delta(\lambda) &= \lambda^2(\lambda+1) \\ \Phi(\lambda) &= \lambda^2(\lambda+1) \end{cases}$$

➡ no internal stability,

➡ try $\mathbf{x}_o$=[0 1 0]′

Jordan Form

$A = QDQ^T$

D	$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$
$\lambda I - D$	$\begin{bmatrix} \lambda-\lambda_1 & 0 & 0 \\ 0 & \lambda-\lambda_1 & 0 \\ 0 & 0 & \lambda-\lambda_1 \end{bmatrix}$	$\begin{bmatrix} \lambda-\lambda_1 & -1 & 0 \\ 0 & \lambda-\lambda_1 & 0 \\ 0 & 0 & \lambda-\lambda_1 \end{bmatrix}$	$\begin{bmatrix} \lambda-\lambda_1 & -1 & 0 \\ 0 & \lambda-\lambda_1 & -1 \\ 0 & 0 & \lambda-\lambda_1 \end{bmatrix}$
$\Delta(\lambda)$	$(\lambda-\lambda_1)(\lambda-\lambda_1)(\lambda-\lambda_1)$		
$D-\lambda_1 I$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
	$(D-\lambda_1 I) = \mathbf{0}$	$(D-\lambda_1 I)^2 = \mathbf{0}$	$(D-\lambda_1 I)^3 = \mathbf{0}$
$\Psi(\lambda)$	$(\lambda-\lambda_1)$	$(\lambda-\lambda_1)^2$	$(\lambda-\lambda_1)^3$

Example 3.11

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix}$$

Asymptotic stability & BIBO stability (5.3)

Corollary:

Asymptotic Stability $\overset{\Rightarrow}{\underset{\Leftarrow}{?}}$ BIBO stability

Zero-input response — Zero-state response

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \qquad \mathbf{u}[k] \equiv 0, \quad \mathbf{x}[0] = \mathbf{x}_0$$

$$\Rightarrow \mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] \qquad \mathbf{x}[0] = \mathbf{x}_0$$

$$\Leftrightarrow \mathbf{x}[k] = \mathbf{A}^k \mathbf{x}_0$$

Definitions are similar to those for continuous-time systems

Theorem 5.D4

1. The equation $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k]$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have magnitudes less than or equal to 1 and those equal to 1 are simple roots of the minimal polynomial of $\mathbf{A}$.

2. The equation $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k]$ is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have magnitudes less than 1.

Proof:

As in continuous-time case,
use equivalence transformation and Jordan form.

Note that:

In discrete-time case,
for certain **A** it is possible that $\mathbf{x}[k] = \mathbf{0}$, $\forall\, k >$ a constant,
which is impossible in continuous-time case.

- Definition:

  A $\in \mathbb{R}^{n\times n}$ is called a stable matrix

  If all eigenvalues of **A** have negative real parts.

From Sec 3.9:

- Positive definiteness of a real symmetric matrix **M**:

$$\mathbf{x}' \mathbf{M} \mathbf{x} > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}.$$

- Positive semi-definiteness of a real symmetric matrix **M**:

$$\mathbf{x}' \mathbf{M} \mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}.$$

Theorem 5.5

All eigenvalues of $\mathbf{A}$ have negative real parts if and only if for any given positive definite symmetric matrix $\mathbf{N}$, the *Lyapunov* equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N} \tag{5.15}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

Theorem 5.5 – 2

Proof:

Theorem 5.5 – 3

Corollary 5.5

All eigenvalues of an $n \times n$ matrix $\mathbf{A}$ have negative real parts if and only if for any given $m \times n$ matrix $\bar{\mathbf{N}}$ with $m < n$ and with the property

$$\operatorname{rank} O := \operatorname{rank} \begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{N}}\mathbf{A} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1} \end{bmatrix} = n \quad \text{(full column rank)} \tag{5.16}$$

where O is an $nm \times n$ matrix, the Lyapunov equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\bar{\mathbf{N}}'\bar{\mathbf{N}} =: -\mathbf{N} \tag{5.17}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

Proof:

Here the main difference is that

N is only positive semidefinite, but rank $O = n$

“⇒”

“⇐”

Theorem 5.6

If all eigenvalues of $\mathbf{A}$ have negative real parts, then the Lyapunov equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

has a unique solution for every $\mathbf{N}$, and the solution can be expressed as

$$\mathbf{M} = \int_0^\infty e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\,dt$$

Proof:

Here no sign definiteness of **N** and **M** is discussed,

so previous proof certainly applies,

but the uniqueness part will be proved again differently

Suppose $\mathbf{M}_1$ and $\mathbf{M}_2$ are two solutions, then

$$\mathbf{A}'(\mathbf{M}_1 - \mathbf{M}_2) + (\mathbf{M}_1 - \mathbf{M}_2)\mathbf{A} = \mathbf{0}$$

➔ $$e^{\mathbf{A}'t}[\mathbf{A}'(\mathbf{M}_1 - \mathbf{M}_2) + (\mathbf{M}_1 - \mathbf{M}_2)\mathbf{A}]e^{\mathbf{A}t} = \frac{d}{dt}[e^{\mathbf{A}'t}(\mathbf{M}_1 - \mathbf{M}_2)e^{\mathbf{A}t}] = \mathbf{0}$$

➔ Integration from 0 to ∞ yields $\quad [e^{\mathbf{A}'t}(\mathbf{M}_1 - \mathbf{M}_2)e^{\mathbf{A}t}]\Big|_0^\infty = \mathbf{0}$

➔ With $e^{\mathbf{A}t} \to \mathbf{0}$ as $t \to \infty$, $\quad \mathbf{0} - (\mathbf{M}_1 - \mathbf{M}_2) = \mathbf{0}$

Note that:

As long as $\lambda_i(\mathbf{A}') + \lambda_j(\mathbf{A}) = \lambda_i(\mathbf{A}) + \lambda_j(\mathbf{A}) \neq 0 \quad \forall i,\ j$

➔ $\forall \mathbf{N}$, $\exists$ a unique solution **M**,

but the integral formula for **M**

does not apply when **A** is not stable,

even if $\lambda_i(\mathbf{A}) + \lambda_j(\mathbf{A}) = 0$ for some (i,j),

solutions **M** may still exist for certain **N**.

$\mathbf{M} - \mathbf{AMB} = \mathbf{C}$, $\quad \mathbf{A}: n \times n, \quad \mathbf{B}: m \times m, \quad \mathbf{M}, \mathbf{C}: n \times m$

$\mathcal{A}(\mathbf{M}) = \mathbf{C}$ where $\mathcal{A}(\mathbf{M}) := \mathbf{M} - \mathbf{AMB}$

: A linear transformation from $R^{n\times m}$ to $R^{n\times m}$

$\mathbf{Au} = \lambda_i \mathbf{u}$: Right eigenvalue-eigenvector pair

$\mathbf{vB} = \mathbf{v}\mu_j$: Left eigenvalue-eigenvector pair

$$\mathcal{A}(\mathbf{uv}) = \mathbf{uv} - \mathbf{AuvB} = (1 - \lambda_i \mu_j)\mathbf{uv}$$

i.e., has eigenvalues

$$\eta_k = 1 - \lambda_i \mu_j \qquad \text{for } i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

For the (discrete-time) Lyapunov equation,

If $\lambda_i \mu_j \neq 1$, $\forall i, j$,

then unique solution **M** exists for all **N**, and

If $\lambda_i \mu_j = 1$ for some (i, j),

then solutions **M** may or may not exist.

Theorem 5.D5

All eigenvalues of an $n \times n$ matrix $\mathbf{A}$ have magnitudes less than 1 if and only if for any given positive definite symmetric matrix $\mathbf{N}$ or for $\mathbf{N} = \bar{\mathbf{N}}'\bar{\mathbf{N}}$, where $\bar{\mathbf{N}}$ is any given $m \times n$ matrix with $m < n$ and with the property in (5.16), the discrete Lyapunov equation

$$\mathbf{M} - \mathbf{A}'\mathbf{M}\mathbf{A} = \mathbf{N} \tag{5.26}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

$$\text{rank } O := \text{rank} \begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{N}}\mathbf{A} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1} \end{bmatrix} = n \quad \text{(full column rank)} \tag{5.16}$$

Proof:

Similar to that for **Corollary 5.5**;
Only sketch of proof is given for the case with **N** > **0** (as in **Thm 5.5**)

"⇒" $|\lambda_i(\mathbf{A})| = |\lambda_i(\mathbf{A}')| < 1$ ➔ unique solution **M** exists for all **N**

$$\mathbf{M} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m \quad \Rightarrow \quad \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m - \mathbf{A}' \left(\sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m \right) \mathbf{A} = \mathbf{N}$$

Also if **N** > **0**, then $\mathbf{x}'\mathbf{M}\mathbf{x} = \mathbf{x}'\mathbf{N}\mathbf{x} + \sum_{m=1}^{\infty} \mathbf{x}'(\mathbf{A}')^m \mathbf{N}\mathbf{A}^m \mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$

"⇐" Let $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$

➔
$$\mathbf{v}^*\mathbf{N}\mathbf{v} = \mathbf{v}^*\mathbf{M}\mathbf{v} - \mathbf{v}^*\mathbf{A}'\mathbf{M}\mathbf{A}\mathbf{v}$$
$$\underbrace{\phantom{\mathbf{v}^*\mathbf{N}\mathbf{v}}}_{>0} = \mathbf{v}^*\mathbf{M}\mathbf{v} - \lambda^*\mathbf{v}^*\mathbf{M}\mathbf{v}\lambda = \underbrace{(1 - |\lambda|^2)}_{>0}\underbrace{\mathbf{v}^*\mathbf{M}\mathbf{v}}_{>0}$$

Theorem 5.D6

If all eigenvalues of **A** have magnitudes less than 1, then the discrete Lyapunov equation

$$\mathbf{M} - \mathbf{A}'\mathbf{M}\mathbf{A} = \mathbf{N}$$

has a unique solution for every **N**, and the solution can be expressed as

$$\mathbf{M} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m$$

Note that:

IF $\lambda_i(\mathbf{A})\lambda_j(\mathbf{A}) \neq 1$, $\forall i, j$

➔ $\forall$**N**, $\exists$ a unique solution **M**,
but the sum formula for **M** applies only when $|\lambda_i(\mathbf{A})| < 1$

IF $\lambda_i(\mathbf{A})\lambda_j(\mathbf{A}) = 1$, for some (i,j), solutions **M** may still exist.

Lyapunov Equation for Discrete-Time Systems

- Relation between C.T. and D.T. Lyapunov Equations

$$s = \frac{z-1}{z+1}, \quad z = \frac{1+s}{1-s}$$

$$\{s \mid \mathrm{Re}(s) < 0\} \longleftrightarrow \{z \mid |z| < 1\}$$

$$\mathbf{M}_d - \mathbf{A}_d'\mathbf{M}_d\mathbf{A}_d = \mathbf{N}_d$$

$$\mathbf{A}_d = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1} \qquad \mathbf{A} = (\mathbf{A}_d+\mathbf{I})^{-1}(\mathbf{A}_d-\mathbf{I})$$

$$\mathbf{A}'\mathbf{M}_d + \mathbf{M}_d\mathbf{A} = -0.5(\mathbf{I}-\mathbf{A}')\mathbf{N}_d(\mathbf{I}-\mathbf{A})$$

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

$$\mathbf{A} = (\mathbf{A}_d-\mathbf{I})(\mathbf{A}_d+\mathbf{I})^{-1} \qquad \mathbf{A}_d = (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})$$

$$\mathbf{M} - \mathbf{A}_d\mathbf{M}\mathbf{A}_d = -0.5(\mathbf{A}_d'+\mathbf{I})\mathbf{N}(\mathbf{A}_d+\mathbf{I})$$

Lyapunov's (First) Stability Theorem

- Theorem 4.1:

  Let $x = 0$ be an equilbrium point for

  $$\dot{x} = f(x) \qquad (4.1)$$

  and $D \subset R^n$ be a domain containing $x = 0$.
  Let $V : D \rightarrow R$ be
  a continuously differentiable function
  such that

  $$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \qquad (4.2)$$

  $$\dot{V}(x) \leq 0 \text{ in } D \qquad (4.3)$$

  Then, $x = 0$ is stable.
  Moreover, if $\dot{V}(x) < 0$ in $D - \{0\}$ (4.4)
  then $x = 0$ is asymptotically stable.

Lyapunov's (First) Stability Theorem – 2

- Briefly Speaking,

Lyapunov's (First) Stability Theorem – 3

- In Linear Case,

2.2: Pendulum Equation w/ Friction – 1

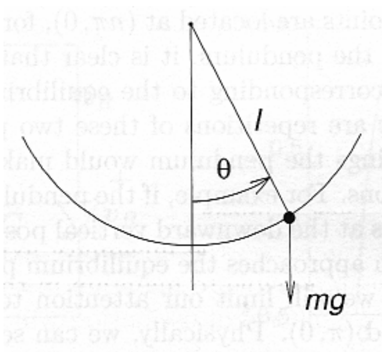
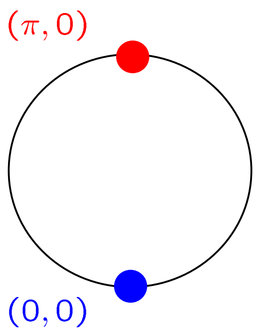


Figure 1.1: Pendulum.



Using Newton's Second Law,
Write the equation of motion
in the tangential direction:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

State model (let $x_1 = \theta, x_2 = \dot{\theta}$):

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

Equilibrium points (let $\dot{x}_1 = \dot{x}_2 = 0$):

$$\begin{aligned}0 &= x_2\\ 0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

Equilibrium points are $(n\pi, 0), n = 0, \pm 1, \pm 2, \ldots$,
or, physically, $(0, 0)$ and $(\pi, 0)$.

Question? Which one is stable or unstable?

Example 4.3: Pendulum without Friction – 1

- Consider the pendulum eqn w/o friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1\end{aligned}$$

and let us study the stability of

the equilibrium point at the origin.

- A natural Lyapunov function candidate

is the energy function

$$V(x) = a(1 - \cos x_1) + (1/2)x_2^2$$

Example 4.3: Pendulum without Friction – 2

$$\dot{V}(x) \quad =$$

Example 4.4: Pendulum with Friction – 1

- Consider the pendulum eqn with friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}$$

- Again, let us try

$$V(x) = a(1 - \cos x_1) + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}b^2 & \frac{1}{2}b \\ \frac{1}{2}b & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 4.4: Pendulum with Friction – 2

$$\dot{V}(x) \quad =$$