Fall 2007

線性系統
Linear Systems

Chapter 06
Controllability & Observability

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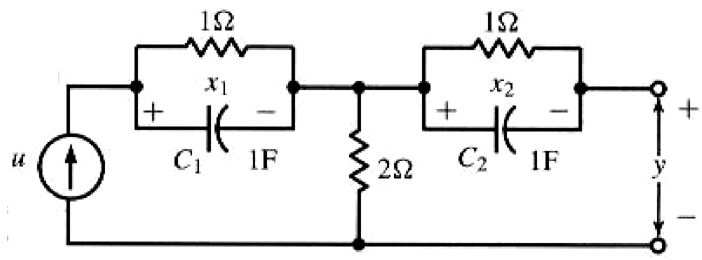
Materials used in these lecture notes are adopted from
"Linear System Theory & Design," 3rd. Ed., by C.-T. Chen (1999)

Outline

- Introduction
- Controllability (6.2)
- Observability (6.3)
- Canonical Decomposition (6.4)
- Conditions in Jordan-Form Equations (6.5)
- Discrete-Time State Equations (6.6)
- Controllability after Sampling (6.7)



1 input: u
2 states: x_1, x_2
1 output: y

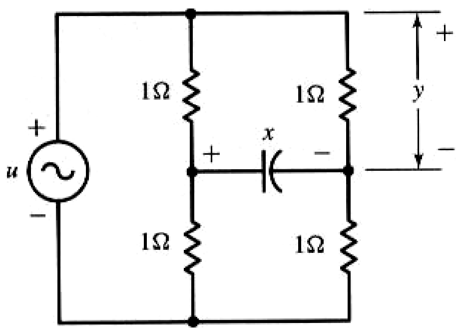
The state x_2 is NOT "controllable" by the input u

The state x_1 is NOT "observable" at the output $y = -x_2 + 2u$

➡ Controllability and observability reveal the internal structure of the system (model)

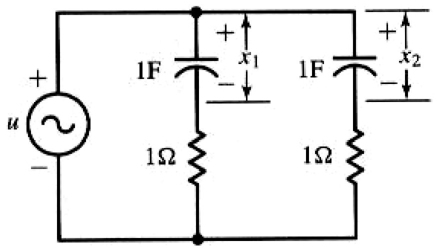
Definition 6.1 $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *or the pair* $(\mathbf{A}, \mathbf{B})$ *is said to be* controllable *if for any initial state* $\mathbf{x}(0) = \mathbf{x}_0$ *and any final state* $\mathbf{x}_1$, *there exists an input that transfers* $\mathbf{x}_0$ *to* $\mathbf{x}_1$ *in a finite time. Otherwise* $(\mathbf{A}, \mathbf{B})$ *is said to be* uncontrollable.

• Un-Controllable Examples:



if $x(0) = 0$,

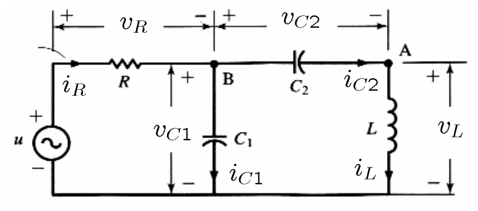
then $x(t) = 0, \quad \forall t \geq 0$,
no matter what $u(t)$ is



if $x_1(0) = x_2(0)$,

then $x_1(t) = x_2(t), \quad \forall t \geq 0$,
no matter what $u(t)$ is

• Controllable Example:



$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

$$\mathbf{A}_p = \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & \frac{3}{5} \\ \frac{5}{3} & -\frac{5}{3} & 0 \end{bmatrix} \quad \mathbf{B}_p = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C}_p = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \mathbf{D}_p = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad \mathbf{C}_c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D}_c = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \mathbf{B}_d = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \quad \mathbf{C}_d = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad \mathbf{D}_d = \begin{bmatrix} 0 \end{bmatrix}$$

Controllability – 4: Diagonal Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= -1\,x_1 + 3\,u \\ \dot{x}_2 &= -2\,x_2 - 6\,u \\ \dot{x}_3 &= -3\,x_3 + 3\,u \\ y &= x_1 \end{aligned}$$

Controllability – 5: Controllable Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6\,x_1 - 11\,x_2 - 6\,x_3 + 6\,u \\ y &= x_1 \end{aligned}$$

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & \frac{3}{5} \\ \frac{5}{3} & -\frac{5}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= -6\,x_1 + -6\,x_3 + 6\,u \\ \dot{x}_2 &= \tfrac{3}{5}\,x_3 \\ \dot{x}_3 &= \tfrac{5}{3}\,x_1 + -\tfrac{5}{3}\,x_2 \\ y &= x_2 \end{aligned}$$

Theorem 6.1

The following statements are equivalent.

1. The n-dimensional pair $(\mathbf{A}, \mathbf{B})$ is controllable.
2. The $n \times n$ matrix

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'\tau}\,d\tau = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'(t-\tau)}\,d\tau$$

   is nonsingular for any $t > 0$.
3. The $n \times np$ *controllability matrix*

$$\mathcal{C} = [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}]$$

   has rank n (full row rank).
4. The $n \times (n + p)$ matrix $[\mathbf{A} - \lambda\mathbf{I}\ \ \mathbf{B}]$ has full row rank at every eigenvalue, λ, of $\mathbf{A}$.[1]
5. If, in addition, all eigenvalues of $\mathbf{A}$ have negative real parts, then the unique solution of

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}' = -\mathbf{B}\mathbf{B}'$$

   is positive definite. The solution is called the *controllability Gramian* and can be expressed as

$$\mathbf{W}_c = \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'\tau}\,d\tau$$

Theorem 6.2 – 2

Proof:

"1. ⇔ 2." "(**A**, **B**) controllable ⇔ $\mathbf{W}_c(t)$ nonsingular, $\forall t > 0$"

Theorem 6.2 – 3

Proof:

Proof:

Proof:

"2. ⇔ 3." "$\mathbf{W}_c(t)$ nonsingular, $\forall t > 0$ ⇔ C has rank n"

Proof:

Proof:

"3. ⇔ 4." "$\mathcal{C}$ has rank n ⇔ rank $[\mathbf{A}-\lambda\mathbf{I}\ \ \mathbf{B}] = n, \forall$ e-value λ of $\mathbf{A}$"

Theorem 6.2 – 8

Proof:

Theorem 6.2 – 9

Proof:

Proof:

"2. ⇔ 5."

"For stable $\mathbf{A}$, $\mathbf{W}_c(t)$ nonsingular, $\forall t > 0$ ⇔
$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}' = -\mathbf{B}\mathbf{B}'$ has a unique P.D. sol. $\mathbf{W}_c(\infty)$"

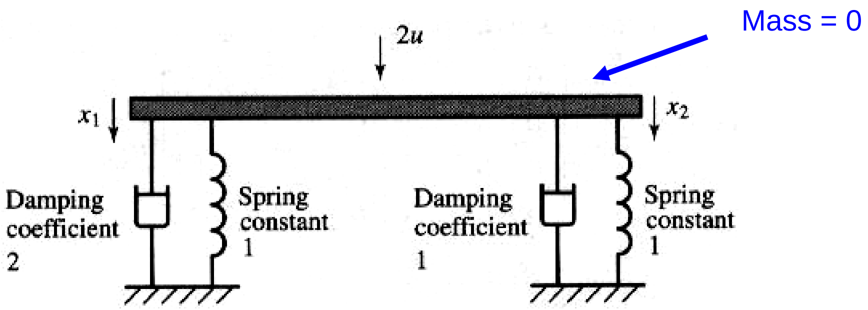
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u$$

$$y = [1 \;\; 0 \;\; 0 \;\; 0]\mathbf{x}$$

$$C = [\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\; \mathbf{A}^3\mathbf{B}] = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}$$

rank = 4

Example 6.3 (6.2)



$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

$$\rho([\mathbf{B} \;\; \mathbf{AB}]) = \rho \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix} = 2$$

Example 6.3 – 2

$$\mathbf{W}_c(t_1) = \int_0^{t_1} e^{\mathbf{A}\,\tau}\, \mathbf{B}\, \mathbf{B}^\top e^{\mathbf{A}^\top \tau}\, d\tau$$

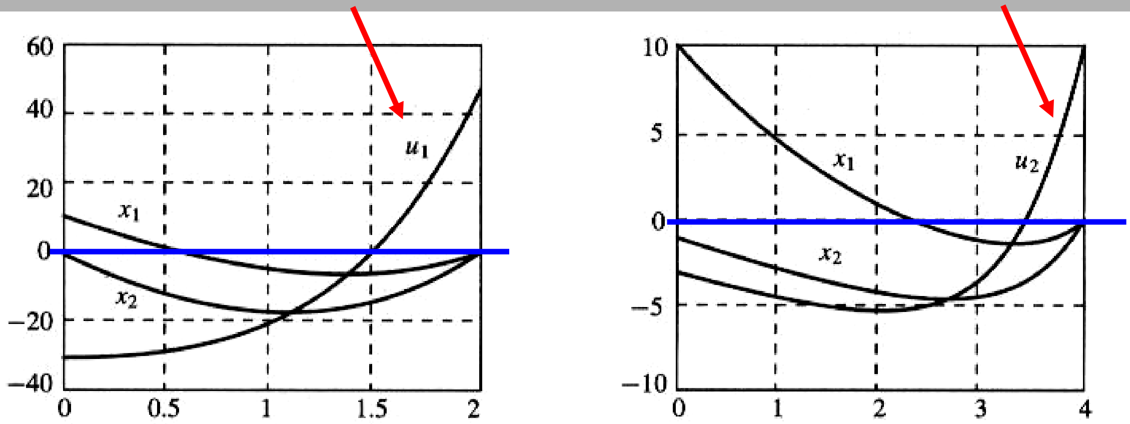
$$\mathbf{W}_c(2) = \int_0^2 \left(\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} [0.5 \;\; 1] \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) d\tau$$

$$= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$$\mathbf{u}(t) = -\mathbf{B}^\top e^{\mathbf{A}^\top (t_1 - t)}\, \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}\, t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

$$u_1(t) = -[0.5 \;\; 1] \begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} \mathbf{W}_c^{-1}(2) \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

$$= -58.82 e^{0.5t} + 27.96 e^{t}$$



"Larger" u_1 transfers $\mathbf{x}(0) = [10\ -1]'$ to $\mathbf{x}(2) = \mathbf{0}$ in 2 seconds, &

"Smaller" u_2 transfers $\mathbf{x}(0) = [10\ -1]'$ to $\mathbf{x}(4) = \mathbf{0}$ in 4 seconds.

Note: Given the same $\mathbf{x}(0)$, t_1, and $\mathbf{x}(t_1)$,
the formula in Theorem 6.1
for $\mathbf{u}(\cdot)$ gives the minimal energy control than other $\bar{\mathbf{u}}(\cdot)$:

$$\int_{t_0}^{t_1} \bar{\mathbf{u}}'(t)\bar{\mathbf{u}}(t)\,dt \geq \int_{t_0}^{t_1} \mathbf{u}'(t)\mathbf{u}(t)\,dt$$

Controllability Index (6.2.1)

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$

$$+ \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$$

$$+ \mathbf{b}_1\, u_1 + \mathbf{b}_2\, u_2 + \cdots + \mathbf{b}_p u_p$$

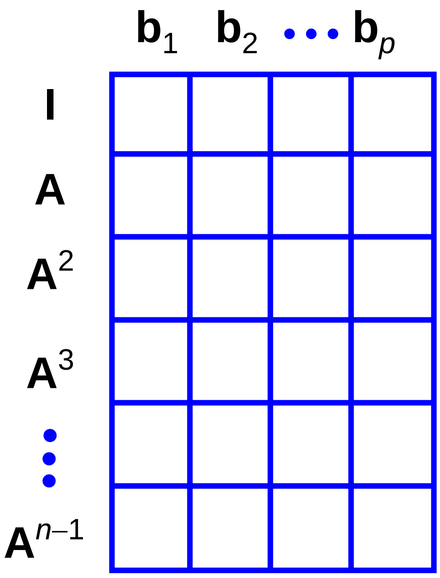
Controllability Index – 2

Given a controllable pair $(\mathbf{A}, \mathbf{B}) \in R^{n\times n} \times R^{n\times p}$ and rank $\mathbf{B} = p$

$$C = [\mathbf{b}_1 \;\; \cdots \;\; \mathbf{b}_p \vdots \mathbf{A}\mathbf{b}_1 \;\; \cdots \;\; \mathbf{A}\mathbf{b}_p \vdots \;\; \cdots \;\; \vdots \mathbf{A}^{n-1}\mathbf{b}_1 \;\; \cdots \;\; \mathbf{A}^{n-1}\mathbf{b}_p]$$

search for n L.I. columns from left to right

Controllability Index – 3



$$\{\mathbf{b}_i, \mathbf{A}\mathbf{b}_i, \mathbf{A}^2\mathbf{b}_i, \; \ldots, \mathbf{A}^{\mu_i - 1}\mathbf{b}_i, \;\; i = 1, 2, \ldots, p\}$$

is a set of n L.I. columns,

and the set

$\{\mu_1, \mu_2, \ldots, \mu_p\}$ with $\mu_1 + \mu_2 + \cdots + \mu_p = n$

is the set of controllability indices

$\mu = \max\{\mu_1, \mu_2, \ldots, \mu_p\}$ is called

the controllability index of $(\mathbf{A}, \mathbf{B})$

$$\boxed{\rho(C_\mu) = \rho([\mathbf{B} \;\; \mathbf{AB} \;\; \cdots \;\; \mathbf{A}^{\mu-1}\mathbf{B}]) = n}$$

If $\mu_1 = \mu_2 = \cdots = \mu_p$, then $n/p = \mu$.

If $\mu_i = 1$ for all $i \neq i_0$, then $\mu = \mu_{i_0} = n - (p - 1)$

If $\bar{n}$ is the degree of the minimal polynomial of $\mathbf{A}$,

then $\mathbf{A}^{\bar{n}} = \tilde{\alpha}_1 \mathbf{A}^{\bar{n}-1} + \tilde{\alpha}_2 \mathbf{A}^{\bar{n}-2} + \cdots + \tilde{\alpha}_{\bar{n}} \mathbf{I}$

and $\mathbf{A}^{\bar{n}}\mathbf{B} = \tilde{\alpha}_1 \mathbf{A}^{\bar{n}-1}\mathbf{B} + \tilde{\alpha}_2 \mathbf{A}^{\bar{n}-2}\mathbf{B} + \cdots + \tilde{\alpha}_{\bar{n}} \mathbf{B}$.

Thus $\mu \leq \bar{n}$.

$$\boxed{n/p \leq \mu \leq \min(\bar{n}, n - p + 1)}$$

Corollary 6.1

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the matrix

$$C_{n-p+1} := [\mathbf{B} \;\; \mathbf{AB} \;\; \cdots \;\; \mathbf{A}^{n-p}\mathbf{B}]$$

where $\rho(\mathbf{B}) = p$, has rank n or the $n \times n$ matrix $C_{n-p+1} C'_{n-p+1}$ is nonsingular.

Example 6.5 (6.2.1)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\Rightarrow \quad [\mathbf{B} \ \ \mathbf{AB} \ \ \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

$$\mu_1 = \mu_2 = \mu = 2$$

Theorem 6.2 (6.2.1)

Theorem 6.2

The controllability property is invariant under any equivalence transformation.

Proof:

Theorem 6.3 (6.2.1)

Theorem 6.3

The set of the controllability indices of $(\mathbf{A}, \mathbf{B})$ is invariant under any equivalence transformation and any reordering of the columns of $\mathbf{B}$.

Proof:

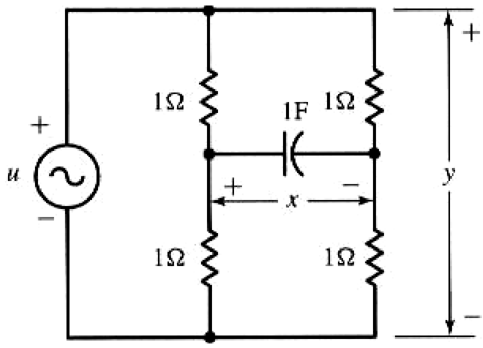
Observability (6.3)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

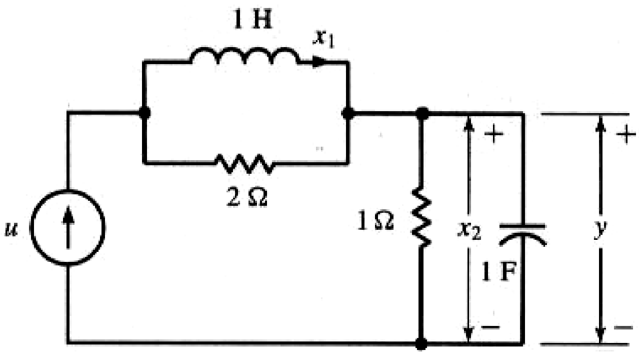
Definition 6.O1 *The state equation* above *is said to be* observable *if for any unknown initial state* $\mathbf{x}(0)$, *there exists a finite* $t_1 > 0$ *such that the knowledge of the input* $\mathbf{u}$ *and the output* $\mathbf{y}$ *over* $[0,\ t_1]$ *suffices to determine uniquely the initial state* $\mathbf{x}(0)$. *Otherwise, the equation is said to be* unobservable.

- Un-Observable Examples:



if $u(t) = 0, \quad \forall t \geq 0,$

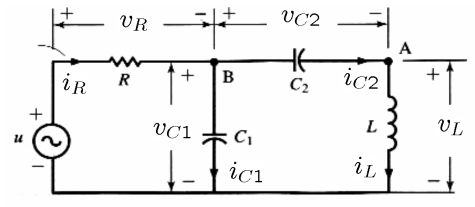
then $y(t) = 0, \quad \forall t \geq 0,$
no matter what $x(0)$ is



if $u(t) = 0, \quad \forall t \geq 0$
and $x_2(0) = 0,$

then $y(t) = 0, \forall t \geq 0,$
no matter what $x_1(0)$ is

- Observable Example:



$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

$$\mathbf{A}_p = \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & \frac{3}{5} \\ \frac{5}{3} & -\frac{5}{3} & 0 \end{bmatrix} \quad \mathbf{B}_p = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C}_p = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \mathbf{D}_p = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad \mathbf{C}_c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D}_c = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \mathbf{B}_d = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \quad \mathbf{C}_d = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad \mathbf{D}_d = \begin{bmatrix} 0 \end{bmatrix}$$

Observability – 4: Diagonal Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= -1\, x_1 + 3\, u \\ \dot{x}_2 &= -2\, x_2 - 6\, u \\ \dot{x}_3 &= -3\, x_3 + 3\, u \\ y &= x_1 \end{aligned}$$

Observability – 5: Controllable Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6\, x_1 - 11\, x_2 - 6\, x_3 + 6\, u \\ y &= x_1 \end{aligned}$$

Observability – 6: Physical Representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & \frac{3}{5} \\ \frac{5}{3} & -\frac{5}{3} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}(t) \end{cases}$$

$$\begin{aligned} \dot{x}_1 &= -6\,x_1 + -6\,x_3 + 6\,u \\ \dot{x}_2 &= \tfrac{3}{5}\,x_3 \\ \dot{x}_3 &= \tfrac{5}{3}\,x_1 + -\tfrac{5}{3}\,x_2 \\ y &= x_2 \end{aligned}$$

Observability – 7

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\boxed{\mathbf{x}(0)} + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t)$$

the only unknown

Re-write:

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \bar{\mathbf{y}}(t) \quad := \underbrace{\mathbf{y}(t) - \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau - \mathbf{D}\mathbf{u}(t)}_{\text{total response – zero-state response}}$$

➡ Observability involves only zero-input response, and is decided by **A** and **C**

$$\underbrace{\mathbf{C}e^{\mathbf{A}t}}_{q\times n \text{ (known)}}\underbrace{\mathbf{x}(0)}_{n\times 1 \text{ (unknown)}} = \underbrace{\bar{\mathbf{y}}(t)}_{q\times 1 \text{ (known)}}$$

: Linear equations

Because $\mathbf{x}(0)$ generates $\bar{\mathbf{y}}(t)$,
the linear equations always have solutions,
and the problem is to determine $\mathbf{x}(0)$ uniquely

For $q < n$, need $\bar{\mathbf{y}}(t)$ at an interval of t
to find the unique solution.

Theorem 6.4

The system (A, B, C, D) is observable if and only if the $n \times n$ matrix

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau$$

is nonsingular for any $t > 0$.

Proof:

"⇐"

Theorem 6.4 – 2

"$\Rightarrow$"

Theorem 6.5 (6.3)

Theorem 6.5 (Theorem of duality)

The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the pair $(\mathbf{A}', \mathbf{B}')$ is observable.

Proof:

Theorem 6.01

The following statements are equivalent.

1. The n-dimensional pair $(\mathbf{A},\ \mathbf{C})$ is observable.
2. The $n \times n$ matrix

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

is nonsingular for any $t > 0$.

3. The $nq \times n$ *observability matrix*

$$O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

has rank n (full column rank). This matrix can be generated by calling `obsv` in MATLAB.

4. The $(n+q) \times n$ matrix

$$\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C} \end{bmatrix}$$

has full column rank at every eigenvalue, λ, of $\mathbf{A}$.

5. If, in addition, all eigenvalues of $\mathbf{A}$ have negative real parts, then the unique solution of

$$\mathbf{A}'\mathbf{W}_o + \mathbf{W}_o\mathbf{A} = -\mathbf{C}'\mathbf{C}$$

is positive definite. The solution is called the *observability Gramian* and can be expressed as

$$\mathbf{W}_o = \int_0^\infty e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

Observability Index (6.3.1)

Given an observable pair $(\mathbf{A}, \mathbf{C}) \in R^{n\times n} \times R^{q\times n}$ and rank $\mathbf{C} = q$

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

$$\{\mathbf{c}_i, \mathbf{c}_i\mathbf{A}, \mathbf{c}_i\mathbf{A}^2, \ \ldots, \mathbf{c}_i\mathbf{A}^{\nu_i - 1}, \ i = 1, 2, \ \ldots, q\}$$

is a set of n L.I. rows, and

the set $\{\nu_1, \nu_2, \ldots, \nu_q\}$ with $\nu_1 + \nu_2 + \cdots + \nu_q = n$

is the set of observability indices

Search for n L.I. rows from top to bottom

$\nu = \max\{\nu_1, \nu_2, \ldots, \nu_q\}$ is called the observability index of $(\mathbf{A}, \mathbf{C})$, and is the least integer such that

$$\rho(O_\nu) := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{\nu-1} \end{bmatrix} = n$$

also, $\boxed{n/q \le \nu \le \min(\bar{n}, n - q + 1)}$

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Corollary 6.O1

The n-dimensional pair $(\mathbf{A}, \mathbf{C})$ is observable if and only if the matrix

$$O_{n-q+1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-q} \end{bmatrix}$$

where $\rho(\mathbf{C}) = q$, has rank n or the $n \times n$ matrix $O'_{n-q+1} O_{n-q+1}$ is nonsingular.

Theorem 6.O2

The observability property is invariant under any equivalence transformation.

Theorem 6.O3

The set of the observability indices of $(\mathbf{A}, \mathbf{C})$ is invariant under any equivalence transformation and any reordering of the rows of $\mathbf{C}$.

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- An Alternative Way to Decide $\mathbf{x}(0)$

Differentiate $\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) = \bar{\mathbf{y}}(t)$ repeatedly and set $t = 0$ to get

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{\nu-1} \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} \bar{\mathbf{y}}(0) \\ \dot{\bar{\mathbf{y}}}(0) \\ \vdots \\ \bar{\mathbf{y}}^{(\nu-1)}(0) \end{bmatrix} \quad \text{or} \quad O_\nu \mathbf{x}(0) = \bar{\mathbf{y}}(0)$$

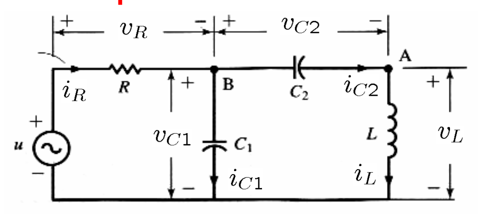
$$O_\nu \mathbf{x}(0) = \tilde{\mathbf{y}}(0)$$

The linear equations have solutions
because $\tilde{\mathbf{y}}(0)$ is generated by $\mathbf{x}(0)$, and have a unique sol.

$$\mathbf{x}(0) = \left[O_\nu' O_\nu\right]^{-1} O_\nu' \tilde{\mathbf{y}}(0)$$

if and only if (**A**, **C**) is observable (rank $O_\nu = n$)
But the method is not very practical,
because derivatives of $\bar{\mathbf{y}}(0)$ are needed

- The Example:



$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

$$\mathbf{A}_p = \begin{bmatrix} -6 & 0 & -6 \\ 0 & 0 & \frac{3}{5} \\ \frac{5}{3} & -\frac{5}{3} & 0 \end{bmatrix} \quad \mathbf{B}_p = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C}_p = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \mathbf{D}_p = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad \mathbf{C}_c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D}_c = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{A}_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \mathbf{B}_d = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \quad \mathbf{C}_d = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad \mathbf{D}_d = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{cases} \dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\,\bar{\mathbf{x}} + \bar{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \bar{\mathbf{C}}\,\bar{\mathbf{x}} + \bar{\mathbf{D}}\,\mathbf{u} \end{cases} \quad \xleftrightarrow{\ \bar{\mathbf{x}} = \mathbf{P}\,\mathbf{x}\ } \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases}$$

$$\begin{aligned} \bar{\mathbf{A}} &= \mathbf{P}\,\mathbf{A}\,\mathbf{P}^{-1} \\ \bar{\mathbf{B}} &= \mathbf{P}\,\mathbf{B} \\ \bar{\mathbf{C}} &= \mathbf{C}\,\mathbf{P}^{-1} \\ \bar{\mathbf{D}} &= \mathbf{D} \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{C}} &= \mathbf{P}\,\mathcal{C} \\ \bar{\mathcal{O}} &= \mathcal{O}\,\mathbf{P}^{-1} \end{aligned}$$

$$\mathbf{x}_c = \mathbf{P}_{cp}\,\mathbf{x}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0.6 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{x}_p$$

$$\mathbf{x}_d = \mathbf{P}_{dp}\,\mathbf{x}_p = \begin{bmatrix} 0.5 & 2.5 & 1.5 \\ -1 & -2 & -2.4 \\ 0.5 & 0.5 & 0.9 \end{bmatrix} \mathbf{x}_p$$

$$\mathbf{x}_c = \mathbf{P}_{cd}\,\mathbf{x}_d = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 0.5 & 4 & 9 \end{bmatrix} \mathbf{x}_d$$

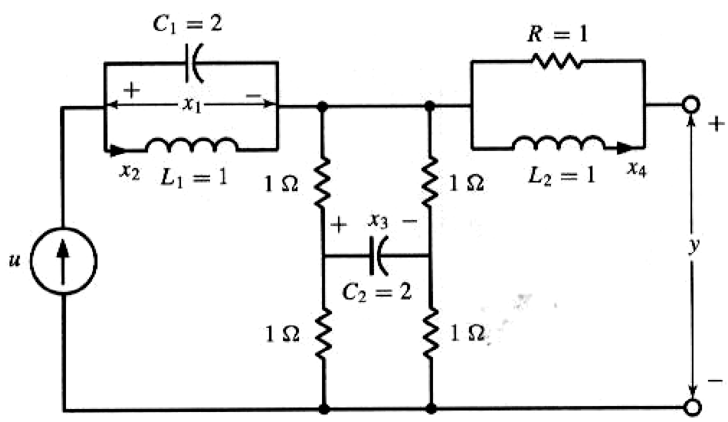
- Stability
- Controllability
- Observability

are preserved

- With appropriate equivalence transformations,

  we may obtain new state equations with following property

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{array}{l} \leftarrow \text{controllable and observable part} \\ \leftarrow \text{controllable and unobservable part} \\ \leftarrow \text{uncontrollable and observable part} \\ \leftarrow \text{uncontrollable and unobservable part} \end{array}$$



Canonical Decomposition – 4

$$\begin{cases} \dot{\bar{\mathrm{x}}} &= \bar{\mathrm{A}}\,\bar{\mathrm{x}} + \bar{\mathrm{B}}\,\mathrm{u} \\ \mathrm{y} &= \bar{\mathrm{C}}\,\bar{\mathrm{x}} + \bar{\mathrm{D}}\,\mathrm{u} \end{cases} \qquad \bar{\mathrm{x}} = \mathrm{P}\,\mathrm{x} \qquad \begin{cases} \dot{\mathrm{x}} &= \mathrm{A}\,\mathrm{x} + \mathrm{B}\,\mathrm{u} \\ \mathrm{y} &= \mathrm{C}\,\mathrm{x} + \mathrm{D}\,\mathrm{u} \end{cases}$$

$$\begin{aligned} \bar{\mathrm{A}} &= \mathrm{P}\,\mathrm{A}\,\mathrm{P}^{-1} \\ \bar{\mathrm{B}} &= \mathrm{P}\,\mathrm{B} \\ \bar{\mathrm{C}} &= \mathrm{C}\,\mathrm{P}^{-1} \\ \bar{\mathrm{D}} &= \mathrm{D} \end{aligned} \qquad \begin{aligned} \bar{\mathcal{C}} &= \mathrm{P}\,\mathcal{C} \\ \bar{\mathcal{O}} &= \mathcal{O}\,\mathrm{P}^{-1} \end{aligned}$$

Theorem 6.6 (6.4)

Theorem 6.6

Consider the n-dimensional state equation **(A, B, C, D)** with

$$\rho(\mathcal{C}) = \rho([\mathbf{B}\ \ \mathbf{AB}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{B}]) = n_1 < n$$

We form the $n \times n$ matrix

$$\mathbf{P}^{-1} := [\mathbf{q}_1\ \ \cdots\ \ \mathbf{q}_{n_1}\ \cdots\ \ \mathbf{q}_n]$$

where the first n_1 columns are any n_1 linearly independent columns of $\mathcal{C}$, and the remaining columns can arbitrarily be chosen as long as $\mathbf{P}$ is nonsingular. Then the equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ or $\mathbf{x} = \mathbf{P}^{-1}\bar{\mathbf{x}}$ will transform **(A, B, C, D)** into

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [\bar{\mathbf{C}}_c\ \ \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

where $\bar{\mathbf{A}}_c$ is $n_1 \times n_1$ and $\bar{\mathbf{A}}_{\bar{c}}$ is $(n - n_1) \times (n - n_1)$, and the n_1-dimensional subequation

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c \mathbf{u}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_c \bar{\mathbf{x}}_c + \mathbf{D}\mathbf{u}$$

is controllable and has the same transfer matrix as **(A, B, C, D)**.

Proof:

$$\{\mathbf{q}_1, \cdots, \mathbf{q}_{n_1}\} \subset \{b_1, b_2, \cdots, b_p, \mathbf{A}b_1, \cdots, \mathbf{A}b_p, \mathbf{A}^2b_1, \cdots, \mathbf{A}^2b_p, \cdots, \mathbf{A}^{n-1}b_p\}$$

$$\text{rank}\left[\mathbf{q}_1 \cdots \mathbf{q}_{n_1}\right] = \text{rank}\left[b_1, b_2, \cdots, b_p, \mathbf{A}b_1, \cdots, \mathbf{A}b_p, \mathbf{A}^2b_1, \cdots, \mathbf{A}^2b_p, \cdots, \mathbf{A}^{n-1}b_p\right]$$

$$\text{span}\left[\mathbf{q}_1 \cdots \mathbf{q}_{n_1}\right] = \text{span}\left[b_1, b_2, \cdots, b_p, \mathbf{A}b_1, \cdots, \mathbf{A}b_p, \mathbf{A}^2b_1, \cdots, \mathbf{A}^2b_p, \cdots, \mathbf{A}^{n-1}b_p\right]$$

$$\{\mathbf{A}\mathbf{q}_1, \cdots, \mathbf{A}\mathbf{q}_{n_1}\} \subset \text{span}\left[\mathbf{q}_1 \cdots \mathbf{q}_{n_1}\right]$$

$$\{\mathbf{q}_{n_1+1}, \cdots, \mathbf{q}_n\} \not\subset \text{span}\left[\mathbf{q}_1 \cdots \mathbf{q}_{n_1}\right]$$

$$\mathbf{A}\mathbf{q}_i = \mathbf{q}_1 + \mathbf{q}_2 + \cdots + \mathbf{q}_{n_1} + \mathbf{q}_{n_1+1} + \cdots + \mathbf{q}_n$$

$$b_i = \mathbf{q}_1 + \mathbf{q}_2 + \cdots + \mathbf{q}_{n_1} + \mathbf{q}_{n_1+1} + \cdots + \mathbf{q}_n$$

Proof:

$$\mathbf{A}\,[\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \mathbf{q}_{n_1+1} \cdots \mathbf{q}_n] = [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \mathbf{q}_{n_1+1} \cdots \mathbf{q}_n]\,\bar{\mathbf{A}}$$

$$= [\mathbf{A}\mathbf{q}_1 \cdots \mathbf{A}\mathbf{q}_{n_1} \mathbf{A}\mathbf{q}_{n_1+1} \cdots \mathbf{A}\mathbf{q}_n] = [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \mathbf{q}_{n_1+1} \cdots \mathbf{q}_n]\begin{bmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{bmatrix}$$

$$\mathbf{B} = [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \mathbf{q}_{n_1+1} \cdots \mathbf{q}_n]\,\bar{\mathbf{B}} = [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \mathbf{q}_{n_1+1} \cdots \mathbf{q}_n]\begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_p]$$

Proof:

$$\bar{\mathbf{C}} = \mathbf{C}\,[\mathbf{q}_1 \;\cdots\; \mathbf{q}_{n_1}\,\mathbf{q}_{n_1+1} \;\cdots\; \mathbf{q}_n] = \begin{bmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * \end{bmatrix}$$

The controllability matrix of the new state equations is

$$\bar{\mathcal{C}} = \begin{bmatrix} \bar{\mathbf{B}}_c & \bar{\mathbf{A}}_c\bar{\mathbf{B}}_c & \cdots & \bar{\mathbf{A}}_c^{n_1}\bar{\mathbf{B}}_c & \cdots & \bar{\mathbf{A}}_c^{n-1}\bar{\mathbf{B}}_c \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Proof:

Thus $\rho(\mathcal{C}) = \rho(\bar{\mathcal{C}}) = n_1$ implies that $(\bar{\mathbf{A}}_c, \bar{\mathbf{B}}_c)$ is controllable

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [\bar{\mathbf{C}}_c \;\; \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transfer Matrix:

$$\mathbf{M} = (s\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}\bar{\mathbf{A}}_{12}(s\mathbf{I} - \bar{\mathbf{A}}_{\bar{c}})^{-1}$$

$$[\bar{\mathbf{C}}_c \;\; \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} s\mathbf{I} - \bar{\mathbf{A}}_c & -\bar{\mathbf{A}}_{12} \\ \mathbf{0} & s\mathbf{I} - \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} + \mathbf{D}$$

$$= [\bar{\mathbf{C}}_c \;\; \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} (s\mathbf{I} - \bar{\mathbf{A}}_c)^{-1} & \mathbf{M} \\ \mathbf{0} & (s\mathbf{I} - \bar{\mathbf{A}}_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} + \mathbf{D}$$

$$= \bar{\mathbf{C}}_c(s\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}\bar{\mathbf{B}}_c + \mathbf{D}$$

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c\bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c\mathbf{u}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_c\bar{\mathbf{x}}_c + \mathbf{D}\mathbf{u}$$

Controllable Realization (6.4)

In the new state equations

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

The state space is divided into a subspace for $\bar{\mathbf{x}}_c$ (dim. = n_1)
And a subspace for $\bar{\mathbf{x}}_{\bar{c}}$ (dim. = $n - n_1$);
$\bar{\mathbf{x}}_c$ is controllable by **u**, while $\bar{\mathbf{x}}_{\bar{c}}$ is not controllable

After dropping the uncontrollable subspace,

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c \mathbf{u}$$

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_c \bar{\mathbf{x}}_c + \mathbf{D}\mathbf{u}$$

becomes a controllable realization of smaller dimension
which is zero-state equivalent to (**A**, **B**, **C**, **D**)

Example 6.8 (6.4)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = [1 \;\; 1 \;\; 1]\mathbf{x}$$

Because rank **B** = 2, use C_2 = [**B** **AB**] to check controllability:

$$\rho(C_2) = \rho([\mathbf{B} \;\; \mathbf{AB}]) = \rho \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 2 < 3$$

: uncontrollable

Choose $\mathbf{P}^{-1} = \mathbf{Q} := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\bar{\mathbf{A}} = \mathbf{PAP}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{PAP}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

$$\bar{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{C}} = \mathbf{CP}^{-1} = [1 \;\; 1 \;\; 1] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [1 \;\; 2 \; \vdots \; 1]$$

A two-dimensional controllable realization:

$$\dot{\bar{\mathbf{x}}}_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bar{\mathbf{x}}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \qquad y = [1 \;\; 2] \bar{\mathbf{x}}_c$$

Theorem 6.O6

Consider the n-dimensional state equation **(A, B, C, D) with**

$$\rho(O) = \rho \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n_2 < n$$

We form the $n \times n$ matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{n_2} \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

where the first n_2 rows are any n_2 linearly independent rows of O, and the remaining rows can be chosen arbitrarily as long as $\mathbf{P}$ is nonsingular. Then the equivalence transformation $\bar{\mathbf{x}} = \mathbf{Px}$ will transform **(A, B, C, D) into**

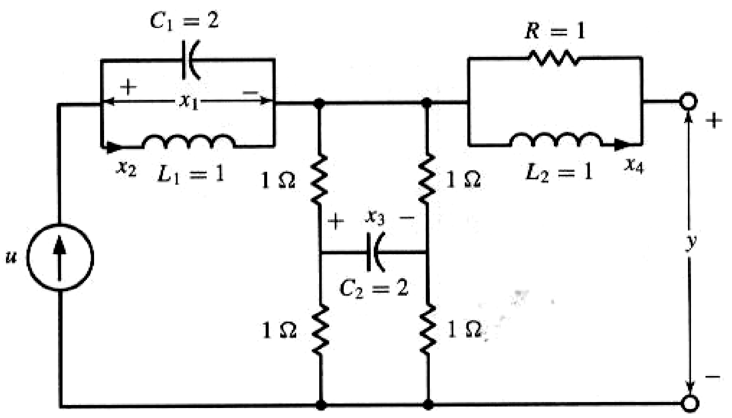
$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [\bar{\mathbf{C}}_o \;\; \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{Du}$$

where $\bar{\mathbf{A}}_o$ is $n_2 \times n_2$ and $\bar{\mathbf{A}}_{\bar{o}}$ is $(n - n_2) \times (n - n_2)$, and the n_2-dimensional subequation

$$\dot{\bar{\mathbf{x}}}_o = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_o + \bar{\mathbf{B}}_o \mathbf{u}$$
$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_o \bar{\mathbf{x}}_o + \mathbf{Du}$$

is observable and has the same transfer matrix as **(A, B, C, D).**



$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix}$$

- $\bar{\mathbf{x}}_{co}$ ← controllable and observable part
- $\bar{\mathbf{x}}_{c\bar{o}}$ ← controllable and unobservable part
- $\bar{\mathbf{x}}_{\bar{c}o}$ ← uncontrollable and observable part
- $\bar{\mathbf{x}}_{\bar{c}\bar{o}}$ ← uncontrollable and unobservable part

Theorem 6.7

Every state-space equation can be transformed, by an equivalence transformation, into the following canonical form

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u} \tag{6.45}$$

$$\mathbf{y} = [\bar{\mathbf{C}}_{co}\ \ \mathbf{0}\ \ \bar{\mathbf{C}}_{\bar{c}o}\ \ \mathbf{0}]\bar{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

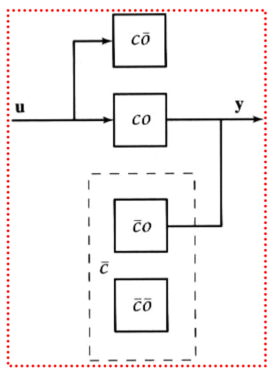
where the vector $\bar{\mathbf{x}}_{co}$ is controllable and observable, $\bar{\mathbf{x}}_{c\bar{o}}$ is controllable but not observable, $\bar{\mathbf{x}}_{\bar{c}o}$ is observable but not controllable, and $\bar{\mathbf{x}}_{\bar{c}\bar{o}}$ is neither controllable nor observable. Furthermore, the state equation is zero-state equivalent to the controllable and observable state equation

$$\dot{\bar{\mathbf{x}}}_{co} = \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u}$$

$$\mathbf{y} = \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u}$$

and has the transfer matrix

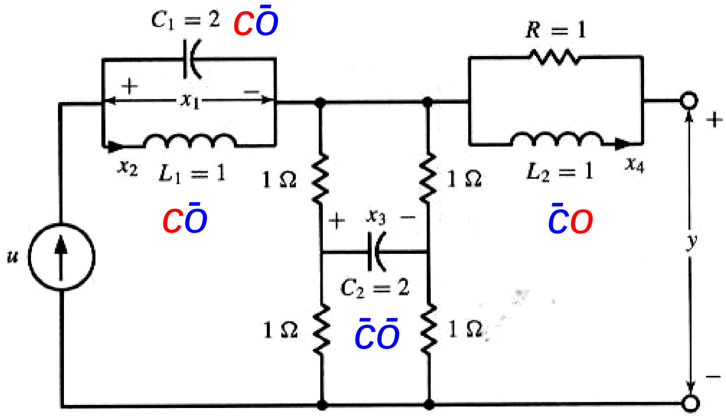
$$\hat{\mathbf{G}}(s) = \bar{\mathbf{C}}_{co}(s\mathbf{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}$$

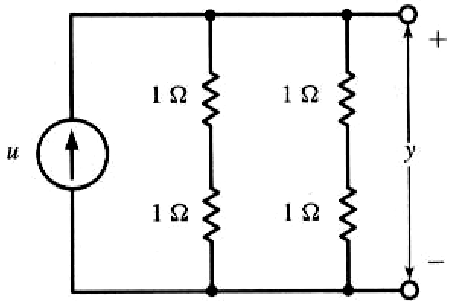


Kalman Decomposition

I/O stability only determined by the controllable and observable parts

Example 6.9 (6.4)





$y = u$

$\hat{g}(s) = 1$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \ \ 0 \ \ 0 \ \ 1]\mathbf{x} + u$$

$$\dot{\mathbf{x}}_c = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = [0 \ \ 0]\mathbf{x}_c + u$$

Conditions in Jordan-Form Equations (6.5)

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{B}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Without loss of generality, consider only the case

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & & & & \mathbf{0} \\ & \mathbf{J}_{12} & & & \\ & & \mathbf{J}_{13} & & \\ & & & \mathbf{J}_{21} & \\ \mathbf{0} & & & & \mathbf{J}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \\ \mathbf{B}_{13} \\ \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{bmatrix}$$

Jordan blocks for λ_1

Jordan blocks for λ_2

$$\mathbf{C} = [\mathbf{C}_{11} \quad \mathbf{C}_{12} \quad \mathbf{C}_{13} \quad \mathbf{C}_{21} \quad \mathbf{C}_{22}]$$

the last row of $\mathbf{B}_{ij}$ is denoted as $\mathbf{b}_{lij}$

the first column of $\mathbf{C}_{ij}$ is denoted as $\mathbf{c}_{fij}$

Theorem 6.8

1. The state equation (**J**, **B**, **C**) is controllable if and only if the three row vectors $\{\mathbf{b}_{l11}, \mathbf{b}_{l12}, \mathbf{b}_{l13}\}$ are linearly independent and the two row vectors $\{\mathbf{b}_{l21}, \mathbf{b}_{l22}\}$ are linearly independent.

2. The state equation (**J**, **B**, **C**) is observable if and only if the three column vectors $\{\mathbf{c}_{f11}, \mathbf{c}_{f12}, \mathbf{c}_{f13}\}$ are linearly independent and the two column vectors $\{\mathbf{c}_{f21}, \mathbf{c}_{f22}\}$ are linearly independent.

Proof:

(for a case where λ_1 has only 2 blocks & λ_2 has only 1 block)

1. Use the controllability condition

rank[**J**–*s***I** **B**] = rank[*s***I**–**J** **B**] = n, for $s = \lambda_1, \lambda_2$.

$$\begin{bmatrix} s-\lambda_1 & -1 & 0 & 0 & 0 & 0 & 0 & \mathbf{b}_{111} \\ 0 & s-\lambda_1 & -1 & 0 & 0 & 0 & 0 & \mathbf{b}_{211} \\ 0 & 0 & s-\lambda_1 & 0 & 0 & 0 & 0 & \mathbf{b}_{l11} \\ 0 & 0 & 0 & s-\lambda_1 & -1 & 0 & 0 & \mathbf{b}_{112} \\ 0 & 0 & 0 & 0 & s-\lambda_1 & 0 & 0 & \mathbf{b}_{l12} \\ 0 & 0 & 0 & 0 & 0 & s-\lambda_2 & -1 & \mathbf{b}_{121} \\ 0 & 0 & 0 & 0 & 0 & 0 & s-\lambda_2 & \mathbf{b}_{l21} \end{bmatrix}$$

Substitute s by λ_1 and get

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \mathbf{b}_{111} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & \mathbf{b}_{211} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{b}_{l11} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & \mathbf{b}_{112} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{b}_{l12} \\ 0 & 0 & 0 & 0 & 0 & \lambda_1-\lambda_2 & -1 & \mathbf{b}_{121} \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1-\lambda_2 & \mathbf{b}_{l21} \end{bmatrix}$$

Examination of the rows reveals that $\mathbf{b}_{l11}$ and $\mathbf{b}_{l12}$ should be L.I. for the matrix to have full row rank.

Similarly, substituting s by λ_2 requires that $\mathbf{b}_{l21}$ be L.I. ($\neq \mathbf{0}$ for one vector).

2. Proof is similar for observability

Example 6.10 (6.5)

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 0 \end{bmatrix} \mathbf{x}$$

Example 6.10 – 2

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{u}$$

L.I.

L.I. ($\neq \mathbf{0}$)

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 0 \end{bmatrix} \mathbf{x}$$

L.I. L.D. ($= \mathbf{0}$)

Corollary 6.8

A single-input Jordan-form state equation is controllable if and only if there is only one Jordan block associated with each distinct eigenvalue and every entry of $\mathbf{B}$ corresponding to the last row of each Jordan block is different from zero.

Corollary 6.O8

A single-output Jordan-form state equation is observable if and only if there is only one Jordan block associated with each distinct eigenvalue and every entry of $\mathbf{C}$ corresponding to the first column of each Jordan block is different from zero.

$$\dot{\mathbf{x}} = \left[\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -2 \end{array}\right]\mathbf{x} + \begin{bmatrix} 10 \\ 9 \\ \boxed{0} \\ \boxed{1} \end{bmatrix} u$$

$$y = [\boxed{1} \; 0 \; 0 \; \boxed{2}]\mathbf{x}$$

Discrete-Time State Equations (6.6): Controllability

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$
$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k]$$

n-dimensional p-input q-output

Definition 6.D1 *The above discrete-time state equation or the pair* $(\mathbf{A}, \mathbf{B})$ *is said to be* controllable *if for any initial state* $\mathbf{x}(0) = \mathbf{x}_0$ *and any final state* $\mathbf{x}_1$*, there exists an input sequence of finite length that transfers* $\mathbf{x}_0$ *to* $\mathbf{x}_1$*. Otherwise the equation or* $(\mathbf{A},\ \mathbf{B})$ *is said to be* uncontrollable.

Theorem 6.D1 (6.6)

Theorem 6.D1

The following statements are equivalent:

1. The n-dimensional pair $(\mathbf{A},\ \mathbf{B})$ is controllable.
2. The $n \times n$ matrix

$$\mathbf{W}_{dc}[n-1] = \sum_{m=0}^{n-1} (\mathbf{A})^m \mathbf{B}\mathbf{B}'(\mathbf{A}')^m$$

   is nonsingular.

3. The $n \times np$ *controllability matrix*

$$\mathcal{C}_d = [\mathbf{B}\ \ \mathbf{A}\mathbf{B}\ \ \mathbf{A}^2\mathbf{B}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{B}]$$

   has rank n (full row rank). The matrix can be generated by calling `ctrb` in MATLAB.

4. The $n \times (n+p)$ matrix $[\mathbf{A} - \lambda\mathbf{I}\ \ \mathbf{B}]$ has full row rank at every eigenvalue, λ, of $\mathbf{A}$.

5. If, in addition, all eigenvalues of $\mathbf{A}$ have magnitudes less than 1, then the unique solution of

$$\mathbf{W}_{dc} - \mathbf{A}\mathbf{W}_{dc}\mathbf{A}' = \mathbf{B}\mathbf{B}'$$

   is positive definite. The solution is called the discrete *controllability Gramian* and can be obtained by using the MATLAB function `dgram`. The discrete Gramian can be expressed as

$$\mathbf{W}_{dc} = \sum_{m=0}^{\infty} \mathbf{A}^m \mathbf{B}\mathbf{B}'(\mathbf{A}')^m$$

Proof:

"1. ⇔ 3." "(**A**, **B**) controllable ⇔ C_d has rank n"

$$\mathbf{x}[n] = \mathbf{A}^n\mathbf{x}[0] + \sum_{m=0}^{n-1}\mathbf{A}^{n-1-m}\mathbf{B}\mathbf{u}[m] \quad \text{i.e.,}$$

$$\underbrace{\mathbf{x}[n] - \mathbf{A}^n\mathbf{x}[0]}_{\text{arbitrary vector}} = \underbrace{[\mathbf{B}\ \ \mathbf{AB}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{B}]}_{C_d \text{ full row rank} \Leftrightarrow \text{input } \mathbf{u}[\cdot] \text{ can always be found}} \begin{bmatrix} \mathbf{u}[n-1] \\ \mathbf{u}[n-2] \\ \vdots \\ \mathbf{u}[0] \end{bmatrix}$$

"2. ⇔ 3." "$\mathbf{W}_{dc}[n-1]$ nonsingular (P.D.) ⇔ C_d has rank n"

$$\mathbf{W}_{dc}[n-1] = [\mathbf{B}\ \ \mathbf{AB}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{B}' \\ \mathbf{B}'\mathbf{A}' \\ \vdots \\ \mathbf{B}'(\mathbf{A}')^{n-1} \end{bmatrix}$$

"3. ⇔ 4." "C_d has rank n ⇔ rank [$\mathbf{A}-\lambda\mathbf{I}$ $\mathbf{B}$] = n, ∀e-value λ of **A**"

The proof is exactly the same as that for the C.T. systems

"2. ⇔ 5." "Suppose **A** has eigenvalues with magnitudes < 1. $\mathbf{W}_{dc}[n-1]$ nonsingular ⇔ $\mathbf{W}_{dc} - \mathbf{A}\mathbf{W}_{dc}\mathbf{A}' = \mathbf{B}\mathbf{B}'$ has a unique positive definite solution $\mathbf{W}_{dc}(\infty)$"

Theorem 5.D6 says that
$\mathbf{W}_{dc} - \mathbf{A}\mathbf{W}_{dc}\mathbf{A}' = \mathbf{B}\mathbf{B}'$ has the unique solution

$$\mathbf{W}_{dc} = \underbrace{\sum_{m=0}^{\infty}\mathbf{A}^m\mathbf{B}\mathbf{B}'(\mathbf{A}')^m}_{>0} = \mathbf{W}_{dc}(\infty) = \underbrace{\mathbf{W}_{dc}[n-1]}_{>0} + \underbrace{\sum_{m=n}^{\infty}\mathbf{A}^m\mathbf{B}\mathbf{B}'(\mathbf{A}')^m}_{\geq 0}$$

Observability (6.6)

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k]$$

n-dimensional p-input q-output

Definition 6.D2 *The above discrete-time state equation or the pair* $(\mathbf{A}, \mathbf{C})$ *is said to be* observable *if for any unknown initial state* $\mathbf{x}[0]$, *there exists a finite integer* $k_1 > 0$ *such that the knowledge of the input sequence* $\mathbf{u}[k]$ *and output sequence* $\mathbf{y}[k]$ *from* $k = 0$ *to* k_1 *suffices to determine uniquely the initial state* $\mathbf{x}[0]$. *Otherwise, the equation is said to be* unobservable.

Theorem 6.DO1 (6.6)

Theorem 6.DO1 (dual to **Theorem 6.D1**)

The following statements are equivalent:

1. The n-dimensional pair $(\mathbf{A}, \mathbf{C})$ is observable.
2. The $n \times n$ matrix

$$\mathbf{W}_{do}[n-1] = \sum_{m=0}^{n-1} (\mathbf{A}')^m \mathbf{C}'\mathbf{C}\mathbf{A}^m$$

   is nonsingular or, equivalently, positive definite.

3. The $nq \times n$ *observability matrix*

$$O_d = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \end{bmatrix}$$

4. The $(n+q) \times n$ matrix

$$\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{B} \end{bmatrix}$$

   has full column rank at every eigenvalue, λ, of $\mathbf{A}$.

5. If, in addition, all eigenvalues of $\mathbf{A}$ have magnitudes less than 1, then the unique solution of

$$\mathbf{W}_{do} - \mathbf{A}'\mathbf{W}_{do}\mathbf{A} = \mathbf{C}'\mathbf{C}$$

   is positive definite. The solution is called the discrete *observability Gramian* and can be expressed as

$$\mathbf{W}_{do} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{C}'\mathbf{C}\mathbf{A}^m$$

Controllability/Observability Indices, Kalman Decomposition, & Jordan-Form Controllability/Observability Conditions for discrete-time systems parallels those for C.T. systems

$$\overbrace{\text{For discrete-time systems,}}$$

Controllability Index =

Length of the shortest input sequence
that can transfer any state to any other state

Observability Index =

Lengths of the shortest input and output sequences
needed to determine the initial state uniquely

In addition to the regular controllability, there are two other "weaker" definitions of controllability:

1. Controllability to the origin:
transfer any state to the zero state;

2. Controllability from the origin:
transfer the zero state to any other state,
also called reachability.

It can be shown that for continuous-time systems, all definitions of controllability are equivalent, but not for discrete-time systems

Controllability to & from the Origin – 2

$$\mathbf{x}[k+1] = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u[k]$$

rank C_d = rank $\begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$ =1:

not controllable, not reachable,

But controllable to the origin:

$u[0] = 2\alpha + \beta$ transfers $\mathbf{x}[0] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to $\mathbf{x}[1] = \mathbf{0}$

Controllability after Sampling (6.7)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{u}(t) = \mathbf{u}(kT) =: \mathbf{u}[k] \qquad \text{for } kT \le t < (k+1)T$$

$$\Downarrow$$

$$\bar{\mathbf{x}}[k+1] = \bar{\mathbf{A}}\bar{\mathbf{x}}[k] + \bar{\mathbf{B}}\mathbf{u}[k]$$

$$\bar{\mathbf{A}} = e^{\mathbf{A}T} \qquad \bar{\mathbf{B}} = \left(\int_0^T e^{\mathbf{A}t} dt \right) \mathbf{B}$$

Theorem 6.9 & 6.10 (6.7)

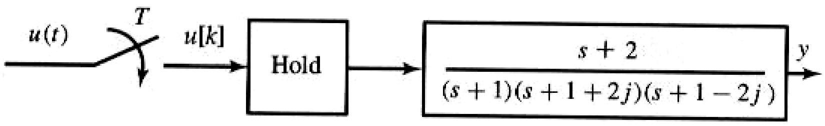
Theorem 6.9

Suppose $(\mathbf{A}, \mathbf{B})$ is controllable. A sufficient condition for its discretized equation $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$, with sampling period T, to be controllable is that $|\mathrm{Im}[\lambda_i - \lambda_j]| \neq 2\pi m/T$ for $m = 1, 2, \ldots,$ whenever $\mathrm{Re}[\lambda_i - \lambda_j] = 0$. For the single-input case, the condition is necessary as well.

Theorem 6.10

If a continuous-time linear time-invariant state equation is not controllable, then its discretized state equation, with any sampling period, is not controllable.

Example 6.12 (6.7)



$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & -7 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \;\; 1 \;\; 2]\mathbf{x}$$

Eigenvalues: –1, $-1 \pm j2$

Discretized systems will be controllable
if and only if the sampling period

$$T \neq \frac{2\pi m}{2} = \pi m \quad \text{and} \quad T \neq \frac{2\pi m}{4} = 0.5\pi m$$ for m = 1, 2, ….

Let us try $T = 0.5\pi$ ($m = 1$):

```
a=[-3 -7 -5;1 0 0;0 1 0];b=[1;0;0];
[ad,bd]=c2d(a,b,pi/2)
```

Example 6.12 – 2

$$\bar{\mathbf{x}}[k+1] = \begin{bmatrix} -0.1039 & 0.2079 & 0.5197 \\ -0.1390 & -0.4158 & -0.5197 \\ 0.1039 & 0.2079 & 0.3118 \end{bmatrix} \bar{\mathbf{x}}[k] + \begin{bmatrix} -0.1039 \\ 0.1039 \\ 0.1376 \end{bmatrix} u[k]$$

$$C_d = \begin{bmatrix} -0.1039 & 0.1039 & -0.0045 \\ 0.1039 & -0.1039 & 0.0045 \\ 0.1376 & 0.0539 & 0.0059 \end{bmatrix}$$

L.D.

and rank $C_d = 2$, uncontrollable