

Fall 2007

線性系統 Linear Systems

Chapter 07 Minimal Realization & Coprime Fractions

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Materials used in these lecture notes are adopted from
“Linear System Theory & Design,” 3rd. Ed., by C.-T. Chen (1999)

Outline

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NTUCEE-LS7-Realization-2

- Introduction (7.1)
- Implications of Coprimeness (7.2)
- Computing Comprime Fractions (7.3)
- Balanced Realization (7.4)
- Degree of Transfer Matrices (7.6)
- Minimal Realizations – Matrix Case (7.7, 7.8, 7.9)

- In Example 4.6:

$$\begin{aligned}\hat{\mathbf{G}}(s) &= \begin{bmatrix} \frac{4s - 10}{2s + 1} & \frac{3}{s+2} \\ 1 & \frac{s+2}{s+1} \\ \hline \frac{(2s+1)(s+2)}{(2s+1)(s+2)} & \frac{(s+2)^2}{(s+2)^2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ 1 & \frac{s+2}{s+1} \\ \hline \frac{(2s+1)(s+2)}{(2s+1)(s+2)} & \frac{(s+2)^2}{(s+2)^2} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{G}}_{sp}(s) &= \frac{1}{s^3 + 4.5s^2 + 6s + 2} \begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix} \\ &= \frac{1}{d(s)} \left(\underbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}_{\mathbf{N}_1} s^2 + \underbrace{\begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}}_{\mathbf{N}_2} s + \underbrace{\begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}}_{\mathbf{N}_3} \right)\end{aligned}$$

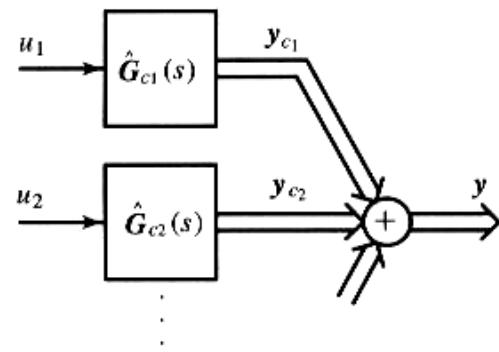
- The first realization is a six-dimensional realization

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1 \mathbf{I}_2 & & \\ & -\alpha_2 \mathbf{I}_2 & \\ & & -\alpha_3 \mathbf{I}_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dots & \dots \\ 0 & 0 \\ 0 & 0 \\ \dots & \dots \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \mathbf{y} &= \begin{bmatrix} -6 & 3 & \vdots & -24 & 7.5 & \vdots & -24 & 3 \\ 0 & 1 & \vdots & 0.5 & 1.5 & \vdots & 1 & 0.5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\end{aligned}$$

$\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3$

- In Example 4.7

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{4s - 10}{2s + 1} & \frac{3}{s+2} \\ 1 & \frac{s+1}{(s+2)^2} \end{bmatrix}$$



$$\dot{\mathbf{x}}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$\mathbf{y}_{c1} = \mathbf{C}_1 \mathbf{x}_1 + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2 \mathbf{x}_2 + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\mathbf{y}_{c2} = \mathbf{C}_2 \mathbf{x}_2 + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2$$

- The second realization is a four-dimensional realization

$$\dot{\mathbf{x}}_1 = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$\mathbf{y}_{c1} = \mathbf{C}_1 \mathbf{x}_1 + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2 \mathbf{x}_2 + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\mathbf{y}_{c2} = \mathbf{C}_2 \mathbf{x}_2 + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2$$

Overall Realization:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \mathbf{y}_{c1} + \mathbf{y}_{c2} = [\mathbf{C}_1 \ \mathbf{C}_2] \mathbf{x} + [\mathbf{d}_1 \ \mathbf{d}_2] \mathbf{u}$$

- The third, fourth, fifth, ... realizations are n_3, n_4, n_5, \dots -dimensional realizations

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{D} \mathbf{u}(t) \end{cases}$$

- The question is

what is the minimal (-dimensional) realization if exists.

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \hat{g}(s) = \frac{N(s)}{D(s)}$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{D} \mathbf{u}(t) \end{cases}$$

$$= \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

Coprimeness of Proper Transfer Functions (SISO) (7.2)

$$\hat{g}(s) = \hat{g}(\infty) + \hat{g}_{sp}(s)$$

Direct transmission part,
“D-matrix” in realization

strictly proper part

Without loss of generality, only discuss

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

$$\hat{y}(s) = N(s) D^{-1}(s) \hat{u}(s)$$

$$\text{Define } \hat{v}(s) = D^{-1}(s) \hat{u}(s)$$

$$\text{Then } \hat{y}(s) = N(s) \hat{v}(s)$$

$$\Rightarrow (s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4) \hat{v}(s) = \hat{u}(s)$$

$$\Rightarrow (b_1 s^3 + b_2 s^2 + b_3 s + b_4) \hat{v}(s) = \hat{y}(s)$$

Define $\mathbf{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} := \begin{bmatrix} \dot{v}(t) \\ \ddot{v}(t) \\ \dot{v}(t) \\ v(t) \end{bmatrix}$

Or $\hat{\mathbf{x}}(s) := \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \hat{x}_3(s) \\ \hat{x}_4(s) \end{bmatrix} := \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \hat{v}(s)$

We have

$$\dot{x}_2 =$$

$$\dot{x}_3 =$$

$$\dot{x}_4 =$$

$$\dot{x}_1 =$$

$$\text{Also, } \hat{y}(s) = N(s) \hat{v}(s) = (b_1 s^3 + b_2 s^2 + b_3 s + b_4) \hat{v}(s)$$

$$= \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \hat{\mathbf{x}}(s)$$

$$y(t) = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \hat{\mathbf{x}}(s)$$

- Thus, a realization is

$$\dot{x} = Ax + bu = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = cx = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} x$$

- Which is called a **controllable canonical form** because

$$C = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 & -a_1^3 + 2a_1a_2 - a_3 \\ 0 & 1 & -a_1 & a_1^2 - a_2 \\ 0 & 0 & 1 & -a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\det(C) = 1$

Theorem 7.1

Theorem 7.1

The controllable canonical form above is observable if and only if $D(s)$ and $N(s)$ in (7.1) are coprime.

Proof:

- More canonical realizations:

$$\hat{g}(s) = \hat{g}^T(s)$$

$$\dot{x} = A^T x + c^T u = \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ -a_4 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u$$

$$y = b^T x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Observable canonical form

- Which may become another **observable canonical form**

$$\left\{ \begin{array}{l} \dot{x} = \begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} x + \begin{bmatrix} b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x \end{array} \right.$$

- And, another **controllable canonical form**

$$\left\{ \begin{array}{l} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} b_4 & b_3 & b_2 & b_1 \end{bmatrix} x \end{array} \right.$$

reverse-labeling x_i 's

- With the equivalence transformation

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Minimal Realizations (7.2.1)

- Realizations with the Lowest Dimension :

- Polynomial fraction: $N(s)/D(s)$

- Coprime fraction: coprime $N(s)/D(s)$

- Characteristic polynomial of $N(s)/D(s)$:

the part of $D(s)$ after the g.c.d. of $N(s)$ and $D(s)$

is factored out

- Degree of $N(s)/D(s)$: degree of its characteristic polynomial

- Example:

$$\frac{s^2 - 1}{4(s^3 - 1)}$$

the coprime fraction:

the characteristic polynomial:

and degree

Theorem 7.2

Theorem 7.2

A state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of a proper rational function $\hat{g}(s)$ if and only if (\mathbf{A}, \mathbf{b}) is controllable and (\mathbf{A}, \mathbf{c}) is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{g}(s)$$

Proof:

1.

$$(a) \left\{ \begin{array}{l} \text{NOT controllable} \\ \text{OR} \\ \text{NOT observable} \end{array} \right. \iff \text{minimal realization}$$

$$(b) \left\{ \begin{array}{l} \text{controllable} \\ \text{AND} \\ \text{observable} \end{array} \right. \iff \text{NOT minimal realization}$$

Theorem 7.2 – 2

Theorem 7.2 – 3

Theorem 7.2 – 4

Theorem 7.2 – 5

2.

Theorem 7.3

All minimal realizations of $\hat{g}(s)$ are equivalent.

Proof:

Implications of Theorems 7.1 - 7.3

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \hat{g}(s) = \frac{N(s)}{D(s)} = c(sI - A)^{-1} b$$

$$\deg \hat{g}(s) =$$

$$\deg D(s) =$$



coprime

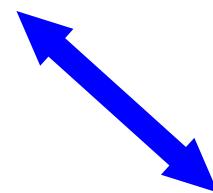
$$\{ A, b, c, d \}$$

$$\dim =$$



$$\{ A_c, b_c, c_c, d_c \}$$

$$\dim =$$



$$\{ A_o, b_o, c_o, d_o \}$$

$$\dim =$$

$\{ \mathbf{A}, \mathbf{b}, \mathbf{c}, d \}$

is a minimal (controllable and observable) realization

$\Leftrightarrow \mathbf{c}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{b}$ is a coprime fraction

$(\Leftrightarrow \dim \mathbf{A} = \deg \hat{g}(s) = \deg \mathbf{c}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{b})$

Controllable/observable canonical forms

based on the coprime fraction of a transfer function

are minimal realizations

Every minimal realization of a T.F. may be transformed to the same controllable/observable canonical forms

Previous definition: pole $p_i : \hat{g}(p_i) = \infty$, zero $z_j : \hat{g}(z_j) = 0$.

\therefore Poles (Zeros) =

roots of denominator (numerator) polynomial of
the coprime fraction of a transfer function

\therefore Poles of a coprime transfer function

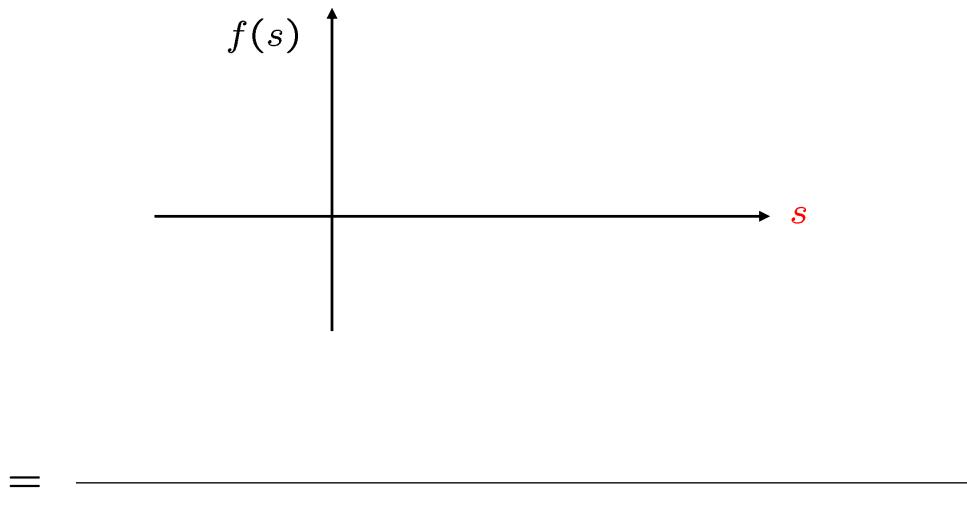
equal to the eigenvalues of the A-matrix

of any minimal realization, and

BIBO stability \Leftrightarrow asymptotic stability

$$\begin{array}{r} 6s^3 + s^2 + 3s - 20 \\ \hline 2s^4 + 7s^3 + 15s^2 + 16s + 10 \end{array}$$

$$f(s) = 2s^4 + 7s^3 + 15s^2 + 16s + 10$$



- Computationally, it is not easy to identify the common roots of $N(s)$ and $D(s)$ for a given $N(s)/D(s)$

- Example:

Roots of $N(s)$: { 0, 15.3, 1×10^8 }

Roots of $D(s)$: { 0.00001, 21.9, 7732.345, 1.00001×10^8 }

- Without loss of generality, consider $N(s)$ and $D(s)$ with

$$\deg N(s) \leq \deg D(s) = n = 4$$

and let

$$\frac{N(s)}{D(s)} = \frac{\bar{N}(s)}{\bar{D}(s)} \quad \text{OR} \quad D(s)(-\bar{N}(s)) + N(s)\bar{D}(s) = 0$$

$N(s)$ and $D(s)$ are not coprime $\iff \exists \bar{N}(s)$ and $\bar{D}(s)$ such that $\deg \bar{N}(s) \leq \deg \bar{D}(s) < n = 4$

$$D(s) = D_0 + D_1s + D_2s^2 + D_3s^3 + D_4s^4 \quad D_4 \neq 0$$

$$N(s) = N_0 + N_1s + N_2s^2 + N_3s^3 + N_4s^4$$

$$\bar{D}(s) = \bar{D}_0 + \bar{D}_1s + \bar{D}_2s^2 + \bar{D}_3s^3$$

$$\bar{N}(s) = \bar{N}_0 + \bar{N}_1s + \bar{N}_2s^2 + \bar{N}_3s^3$$

$$D(s)(-\bar{N}(s)) + N(s)\bar{D}(s) = 0 \iff$$

$$\begin{bmatrix} D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 \\ D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 \\ D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 \\ 0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4 \end{bmatrix} \begin{bmatrix} -\bar{N}_0 \\ \bar{D}_0 \\ \dots \\ -\bar{N}_1 \\ \bar{D}_1 \\ \dots \\ -\bar{N}_2 \\ \bar{D}_2 \\ \dots \\ -\bar{N}_3 \\ \bar{D}_3 \end{bmatrix} = \mathbf{0}$$

Sylvester Resultant \mathbf{S} : $2n \times 2n$

$N(s)$ and $D(s)$ are coprime $\iff \mathbf{S}$ is nonsingular

- When $N(s)$ and $D(s)$ are not coprime,

Search the L.I. columns of \mathbf{S} from left to right:

D -columns, all L.I., since $D_4 \neq 0$

$$\left[\begin{array}{cc|cc|cc|cc} D_0 & N_0 & : & 0 & 0 & : & 0 & 0 & : & 0 & 0 \\ D_1 & N_1 & : & D_0 & N_0 & : & 0 & 0 & : & 0 & 0 \\ D_2 & N_2 & : & D_1 & N_1 & : & D_0 & N_0 & : & 0 & 0 \\ D_3 & N_3 & : & D_2 & N_2 & : & D_1 & N_1 & : & D_0 & N_0 \\ D_4 & N_4 & : & D_3 & N_3 & : & D_2 & N_2 & : & D_1 & N_1 \\ 0 & 0 & : & D_4 & N_4 & : & D_3 & N_3 & : & D_2 & N_2 \\ 0 & 0 & : & 0 & 0 & : & D_4 & N_4 & : & D_3 & N_3 \\ 0 & 0 & : & 0 & 0 & : & 0 & 0 & : & D_4 & N_4 \end{array} \right]$$

If an N -column linearly depends on its LHS columns, then all subsequent N -columns depend on their LHS columns

The first L.D. N -column is called the primary dependent N -column

N -columns, may be L.D.

Thus, $\deg N(s)/D(s) = \mu$, the number of L.I. N -columns, and $\text{rank } \mathbf{S} = n + \mu$ for non-coprime $N(s)$ and $D(s)$

$$\left[\begin{array}{cc|cc|cc|cc} D_0 & N_0 & : & 0 & 0 & : & 0 & 0 & : & 0 & 0 \\ D_1 & N_1 & : & D_0 & N_0 & : & 0 & 0 & : & 0 & 0 \\ D_2 & N_2 & : & D_1 & N_1 & : & D_0 & N_0 & : & 0 & 0 \\ D_3 & N_3 & : & D_2 & N_2 & : & D_1 & N_1 & : & D_0 & N_0 \\ D_4 & N_4 & : & D_3 & N_3 & : & D_2 & N_2 & : & D_1 & N_1 \\ 0 & 0 & : & D_4 & N_4 & : & D_3 & N_3 & : & D_2 & N_2 \\ 0 & 0 & : & 0 & 0 & : & D_4 & N_4 & : & D_3 & N_3 \\ 0 & 0 & : & 0 & 0 & : & 0 & 0 & : & D_4 & N_4 \end{array} \right] \left[\begin{array}{c} -\bar{N}_0 \\ \bar{D}_0 \\ \dots \\ -\bar{N}_1 \\ \bar{D}_1 \\ \dots \\ -\bar{N}_2 \\ \bar{D}_2 \\ \dots \\ -\bar{N}_3 \\ \bar{D}_3 \end{array} \right] = \mathbf{0}$$

$\mathbf{S}_1 : 2n \times 2(\mu+1)$,

$\text{rank } \mathbf{S}_1 = 2\mu + 1$

Example 7.1

$$\frac{N(s)}{D(s)} = \frac{6s^3 + s^2 + 3s - 20}{2s^4 + 7s^3 + 15s^2 + 16s + 10}$$

$$\left[\begin{array}{cccccccc} 10 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 3 & 10 & -20 & 0 & 0 & 0 & 0 \\ 15 & 1 & 16 & 3 & 10 & -20 & 0 & 0 \\ 7 & 6 & 15 & 1 & 16 & 3 & 10 & -20 \\ 2 & 0 & 7 & 6 & 15 & 1 & 16 & 3 \\ 0 & 0 & 2 & 0 & 7 & 6 & 15 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]$$

Theorem 7.4

Theorem 7.4

Consider $\hat{g}(s) = N(s)/D(s)$. We use the coefficients of $D(s)$ and $N(s)$ to form the Sylvester resultant \mathbf{S} in (7.28) and search its linearly independent columns in order from left to right. Then we have

$$\deg \hat{g}(s) = \text{number of linearly independent } N\text{-columns} =: \mu$$

and the coefficients of a coprime fraction $\hat{g}(s) = \bar{N}(s)/\bar{D}(s)$ or

$$[-\bar{N}_0 \quad \bar{D}_0 \quad -\bar{N}_1 \quad \bar{D}_1 \quad \cdots \quad -\bar{N}_\mu \quad \bar{D}_\mu]'$$

equals the monic null vector of the submatrix that consists of the primary dependent N -column and all its LHS linearly independent columns of \mathbf{S} .

- A good numerical algorithm for searching L.I. columns of \mathbf{S}

— The QR decomposition (details omitted)

Given $\mathbf{M} \in \mathbb{R}^{n \times m}$

Exist an orthogonal $\bar{\mathbf{Q}} \in \mathbb{R}^{n \times n}$

$$\bar{\mathbf{Q}}^{-1} = \bar{\mathbf{Q}}^T =: Q$$

$$\bar{\mathbf{Q}}\mathbf{M} = \mathbf{R}$$

$$\mathbf{M} = \mathbf{Q}\mathbf{R}$$

Example 7.1

$$\frac{N(s)}{D(s)} = \frac{6s^3 + s^2 + 3s - 20}{2s^4 + 7s^3 + 15s^2 + 16s + 10}$$

$$\left[\begin{array}{cccccccc} 10 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 3 & 10 & -20 & 0 & 0 & 0 & 0 \\ 15 & 1 & 16 & 3 & 10 & -20 & 0 & 0 \\ 7 & 6 & 15 & 1 & 16 & 3 & 10 & -20 \\ 2 & 0 & 7 & 6 & 15 & 1 & 16 & 3 \\ 0 & 0 & 2 & 0 & 7 & 6 & 15 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]$$

In Matlab: $[Q, R] = qr(S)$

$$R = \left[\begin{array}{cccccccc} -25.1 & 3.7 & -20.6 & 10.1 & -11.6 & 11.0 & -4.1 & 5.3 \\ 0 & -20.7 & -10.3 & 4.3 & -7.2 & 2.1 & -3.6 & 6.7 \\ 0 & 0 & -10.2 & -15.6 & -20.3 & 0.8 & -16.8 & 9.6 \\ 0 & 0 & 0 & 8.9 & -3.5 & -17.9 & -11.2 & 7.3 \\ 0 & 0 & 0 & 0 & -5.0 & 0 & -12.0 & -15.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

• Example

$$\dot{x} = \begin{bmatrix} -1 & \frac{-4}{\alpha} \\ 4\alpha & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2\alpha \end{bmatrix} u \quad \alpha \neq 0$$

$$y = \begin{bmatrix} -1 & \frac{2}{\alpha} \end{bmatrix} x$$

$$\text{transfer function } \hat{g}(s) = c(sI - A)^{-1} b = \frac{3s + 18}{s^2 + 3s + 18}$$

$$\text{controllability matrix } C = [b \ Ab] = \begin{bmatrix} 1 & -9 \\ 2\alpha & 0 \end{bmatrix}$$

$$\text{observability matrix } O = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{\alpha} \\ 9 & 0 \end{bmatrix}$$

→ The above realization is _____ and _____.

• Since A is stable,

the solutions of the following equations are positive definite

$$A W_c + W_c A^\top = -b b^\top$$

$$A^\top W_o + W_o A = -c^\top c$$

$$\text{controllability Gramian } W_c = \begin{bmatrix} 0.5 & 0 \\ 0 & \alpha^2 \end{bmatrix}$$

$$\text{observability Gramian } W_o = \begin{bmatrix} 0.5 & 0 \\ 0 & \frac{1}{\alpha^2} \end{bmatrix}$$

→ $\alpha > 1$: “more controllable”, and $\alpha < 1$: “more observable”

Also,

$$\mathbf{W}_c \mathbf{W}_o = \begin{bmatrix} 0.5 & 0 \\ 0 & \alpha^2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & \frac{1}{\alpha^2} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

For the general case,

is there a minimal realization

which is “balanced” in controllability and observability?

Theorem 7.5

Theorem 7.5

Let (A, b, c) and $(\bar{A}, \bar{b}, \bar{c})$ be minimal and equivalent and let $\mathbf{W}_c \mathbf{W}_o$ and $\bar{\mathbf{W}}_c \bar{\mathbf{W}}_o$ be the products of their controllability and observability Gramians. Then $\mathbf{W}_c \mathbf{W}_o$ and $\bar{\mathbf{W}}_c \bar{\mathbf{W}}_o$ are similar and their eigenvalues are all real and positive.

Proof:

$$A \mathbf{W}_c + \mathbf{W}_c A^\top = -b b^\top$$

$$A^\top \mathbf{W}_o + \mathbf{W}_o A = -c^\top c$$

$$\bar{A} \bar{\mathbf{W}}_c + \bar{\mathbf{W}}_c \bar{A}^\top = -\bar{b} \bar{b}^\top$$

$$\bar{A}^\top \bar{\mathbf{W}}_o + \bar{\mathbf{W}}_o \bar{A} = -\bar{c}^\top \bar{c}$$

• Corollary:

For any **stable** realization,

$\mathbf{W}_c \mathbf{W}_o$ is **similar** to Σ^2 ,

where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$

and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$

are **positive square roots** of the **eigenvalues** of $\mathbf{W}_c \mathbf{W}_o$,

and these **eigenvalues** are called the **Hankel singular values**

Theorem 7.6 (The balanced realization)

For any n -dimensional minimal state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, there exists an equivalence transformation $\bar{\mathbf{x}} = \mathbf{Px}$ such that the controllability Gramian $\bar{\mathbf{W}}_c$ and observability Gramian $\bar{\mathbf{W}}_o$ of its equivalent state equation have the property

$$\bar{\mathbf{W}}_c = \bar{\mathbf{W}}_o = \Sigma$$

Proof:

It is also possible to find \mathbf{P} such that

$\bar{\mathbf{W}}_c = \mathbf{I}$ and $\bar{\mathbf{W}}_o = \Sigma^2$ (**input-normal realization**), or such that

$\bar{\mathbf{W}}_o = \mathbf{I}$ and $\bar{\mathbf{W}}_c = \Sigma^2$ (**output-normal realization**).

Suppose $\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} u$

$$y = [\mathbf{c}_1 \quad \mathbf{c}_2] \mathbf{x}$$

is a **balanced minimal** realization of a **stable** $\hat{g}(s)$ with

$$\mathbf{W}_c = \mathbf{W}_o = \text{diag}(\Sigma_1, \Sigma_2)$$

If the **Hankel singular values** σ_i in Σ_1 and Σ_2 are different, then

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11} \mathbf{x}_1 + \mathbf{b}_1 u$$

$$y = \mathbf{c}_1 \mathbf{x}_1$$

is **balanced** and \mathbf{A}_{11} is **stable**, and

If the σ_i in Σ_1 are much larger than those in Σ_2 ,

Then $\mathbf{c}_1(sI - \mathbf{A}_{11})^{-1} \mathbf{b}_1 \approx \hat{g}(s)$

System Reduction

- For MIMO cases:

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

Definition 7.1 The characteristic polynomial of a proper rational matrix $\hat{\mathbf{G}}(s)$ is defined as the least common denominator of all minors of $\hat{\mathbf{G}}(s)$. The degree of the characteristic polynomial is defined as the McMillan degree or, simply, the degree of $\hat{\mathbf{G}}(s)$ and is denoted by $\delta\hat{\mathbf{G}}(s)$.

$$\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \quad \hat{G}_2(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Minor of order 1:

Minor of order 2:

→ Characteristic polynomial:

Degree:

Minor of order 1:

Minor of order 2:

→ Characteristic polynomial:

Degree:

Characteristic polynomial of a transfer matrix is in general different from the least common denominator of all its elements

Example 7.5

$$\hat{G}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

Minors of order 1: all six elements

Minors of order 2:

$$\begin{aligned} \frac{s}{(s+1)^2(s+2)} + \frac{1}{(s+1)^2(s+2)} &= \frac{s+1}{(s+1)^2(s+2)} = \frac{1}{(s+1)(s+2)} \\ \frac{s}{s+1} \cdot \frac{1}{s} + \frac{1}{(s+1)(s+3)} &= \frac{s+4}{(s+1)(s+3)} \\ \frac{1}{(s+1)(s+2)s} - \frac{1}{(s+1)(s+2)(s+3)} &= \frac{3}{s(s+1)(s+2)(s+3)} \end{aligned}$$

Characteristic polynomial: $s(s+1)(s+2)(s+3)$; Degree: 4

$$\hat{G}(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)} & \frac{s+2}{s^2} \\ \frac{s-1}{s+3} & \frac{s}{(s+5)(s-3)} \end{bmatrix}$$

Every element has different poles

→ Characteristic polynomial equals

the product of the denominators of all elements

Let (A, B, C, D) be a controllable and observable realization of $\hat{G}(s)$:

- Monic least common denominator of all minors of $\hat{G}(s)$

= Characteristic polynomial of A

- Monic least common denominator of all entries of $\hat{G}(s)$

= Minimal polynomial of A

Theorem 7.M2

A state equation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization of a proper rational matrix $\hat{\mathbf{G}}(s)$ if and only if (\mathbf{A}, \mathbf{B}) is controllable and (\mathbf{A}, \mathbf{C}) is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{\mathbf{G}}(s)$$

Theorem 7.M2 – 2**Proof:**

For “minimality” \Leftrightarrow controllability and observability only

“ \Rightarrow ” If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is not controllable OR not observable,

Then can find a reduced dimensional realization (Thm 6.7)

Thus $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is not minimal

Proof:

“ \Leftarrow ” If (A, B, C, D) is n -dimensional, controllable & observable,

BUT there is another \bar{n} -dimensional realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$

with $\bar{n} < n$, then $CA^m B = \bar{C}\bar{A}^m \bar{B}$ for $m = 0, 1, 2, \dots$

i.e.,
$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \underbrace{\left[B \ AB \ \dots \ A^{n-1}B \right]}_{\text{rank} \geq n + n - n = n} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} \underbrace{\left[\bar{B} \ \bar{A}\bar{B} \ \dots \ \bar{A}^{n-1}\bar{B} \right]}_{\text{rank} \leq n < n}$$

**Theorem 7.M3**

All minimal realizations of $\hat{G}(s)$ are equivalent.

Proof:

Consider any two n -dimensional **minimal** realizations

(A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, thus,

$$\mathcal{O}C = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \left[B \ AB \ \dots \ A^{n-1}B \right] = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} \left[\bar{B} \ \bar{A}\bar{B} \ \dots \ \bar{A}^{n-1}\bar{B} \right] = \bar{\mathcal{O}}\bar{C}$$

and $\mathcal{O}AC = \bar{\mathcal{O}}\bar{A}\bar{C}$

Theorem 7.M3 – 2

Let $\begin{cases} \mathcal{O}^+ := (\mathcal{O}^\top \mathcal{O})^{-1} \mathcal{O}^\top \\ \mathcal{C}^+ := \mathcal{C}^\top (\mathcal{C} \mathcal{C}^\top)^{-1} \\ \mathbf{P} := \bar{\mathcal{O}}^+ \mathcal{O} = (\bar{\mathcal{O}}^\top \bar{\mathcal{O}})^{-1} \bar{\mathcal{O}}^\top \mathcal{O} \\ = \bar{\mathcal{C}} \mathcal{C}^+ = \bar{\mathcal{C}} \mathcal{C}^\top (\mathcal{C} \mathcal{C}^\top)^{-1} \end{cases}$ Then $\begin{cases} \mathcal{O}^+ \mathcal{O} = (\mathcal{O}^\top \mathcal{O})^{-1} \mathcal{O}^\top \mathcal{O} = \mathbf{I} \\ \mathcal{C} \mathcal{C}^+ = \mathcal{C} \mathcal{C}^\top (\mathcal{C} \mathcal{C}^\top)^{-1} = \mathbf{I} \\ \mathbf{P}^{-1} := \mathcal{O}^+ \bar{\mathcal{O}} = (\mathcal{O}^\top \mathcal{O})^{-1} \mathcal{O}^\top \bar{\mathcal{O}} \\ = \mathcal{C} \bar{\mathcal{C}}^+ = \mathcal{C} \bar{\mathcal{C}}^\top (\bar{\mathcal{C}} \bar{\mathcal{C}}^\top)^{-1} \end{cases}$

Therefore, from $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$ we have

$$\bar{\mathcal{C}} = \bar{\mathcal{O}}^+ \mathcal{O}\mathcal{C} = (\bar{\mathcal{O}}^\top \bar{\mathcal{O}})^{-1} \bar{\mathcal{O}}^\top \mathcal{O}\mathcal{C} = \mathbf{P}\mathcal{C}$$

$$\bar{\mathcal{O}} = \mathcal{O}\mathcal{C}\bar{\mathcal{C}}^+ = \mathcal{O}\mathcal{C}\bar{\mathcal{C}}^\top (\bar{\mathcal{C}}\bar{\mathcal{C}}^\top)^{-1} = \mathcal{O}\mathbf{P}^{-1}$$

which imply $\bar{\mathbf{B}} = \mathbf{PB}$ and $\bar{\mathbf{C}} = \mathbf{CP}^{-1}$

Also, from $\bar{\mathcal{O}}\bar{\mathbf{A}}\bar{\mathcal{C}} = \mathcal{O}\mathbf{A}\mathcal{C}$ we have

$$\bar{\mathbf{A}} = (\bar{\mathcal{O}}^\top \bar{\mathcal{O}})^{-1} \bar{\mathcal{O}}^\top \mathcal{O}\mathbf{A}\mathcal{C}\bar{\mathcal{C}}^\top (\bar{\mathcal{C}}\bar{\mathcal{C}}^\top)^{-1} = \mathbf{PAP}^{-1}$$

Example 7.6

• In Example 4.6:

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{4s - 10}{2s + 1} & \frac{3}{s + 2} \\ 1 & \frac{1}{(s + 2)^2} \end{bmatrix}$$

Characteristic polynomial: $(2s+1)(s+2)^2$; Degree: 3

In Matlab: [Am, Bm, Cm, Dm] = minreal(A, B, C, D)

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.8625 & -4.0897 & 3.2544 \\ 0.2921 & -3.0508 & 1.2709 \\ -0.0944 & 0.3377 & -0.5867 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.3218 & -0.5305 \\ 0.0459 & -0.4983 \\ -0.1688 & 0.0840 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & -0.0339 & 35.5281 \\ 0 & -2.1031 & -0.5720 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

a minimal (controllable and observable) realization

• Example 7.7:

$$\hat{g}(s) = \frac{n(s)}{d(s)} =$$

$$\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix} =$$

$$\hat{G}(s) = \begin{bmatrix} (2s+1)(s+2) & 0 \\ 0 & (2s+1)(s+2)^2 \end{bmatrix}^{-1} \times \begin{bmatrix} (4s-10)(s+2) & 3(2s+1) \\ (s+2) & (s+1)(2s+1) \end{bmatrix}$$

$$\hat{G}(s) = \begin{bmatrix} (2s-5)(s+2) & (4s-7) \\ 0.5 & 1 \end{bmatrix} \times \begin{bmatrix} (s+2)(s+0.5) & (2s+1) \\ 0 & (s+2) \end{bmatrix}^{-1}$$

Example 7.8

$$\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\frac{12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

$$\hat{G}_{sp}(s) = \begin{bmatrix} -6s-12 & -9 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} s^2+2.5s+1 & 2s+1 \\ 0 & s+2 \end{bmatrix}^{-1}$$

$$H(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \quad D(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} H(s) + \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} L(s)$$

$$L(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N(s) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 0.5 & 1 \end{bmatrix} L(s)$$

Example 7.8 – 2

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{H}(s) + \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{L}(s)$$

$$\mathbf{D}_{hc}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \mathbf{H}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}$$

$$\mathbf{D}_{hc}^{-1} \mathbf{D}_{lc} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2.5 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$s^2 + 2.5s + 1 \quad \ddot{x} + 2.5\dot{x} + x$$

$$x_1 = \dot{x}$$

$$x_2 = x$$

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dots \end{bmatrix} \mathbf{x} + \dots$$

Example 7.8 – 3

$$\mathbf{H}(s) = -\begin{bmatrix} 2.5 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{L}(s) + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \mathbf{D}(s)$$

$$\mathbf{N}(s) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 0.5 & 1 \end{bmatrix} \mathbf{L}(s)$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2.5 & -1 & \vdots & 3 \\ 1 & 0 & \vdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \vdots & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ \dots & \dots \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} -6 & -12 & \vdots & -9 \\ 0 & 0.5 & \vdots & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

Particular Example (7.9)

$$\mathbf{H}(s) = \begin{bmatrix} s^4 & 0 \\ 0 & s^3 \end{bmatrix}$$

$$\mathbf{L}(s) = \begin{bmatrix} s^3 & 0 \\ s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)$$

$$\mathbf{D}_{hc}^{-1} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc} = \begin{bmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{21}^1 & a_{22}^1 & a_{23}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{21}^2 & a_{22}^2 & a_{23}^2 \end{bmatrix} \mathbf{L}(s)$$

$$\mathbf{N}(s) = \begin{bmatrix} b_{11}^1 & b_{12}^1 & b_{13}^1 & b_{14}^1 & b_{21}^1 & b_{22}^1 & b_{23}^1 \\ b_{11}^2 & b_{12}^2 & b_{13}^2 & b_{14}^2 & b_{21}^2 & b_{22}^2 & b_{23}^2 \end{bmatrix} \mathbf{L}(s)$$

Particular Example – 2

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_{11}^1 & -a_{12}^1 & -a_{13}^1 & -a_{14}^1 & \vdots & -a_{21}^1 & -a_{22}^1 & -a_{23}^1 \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ \dots & \dots \\ -a_{11}^2 & -a_{12}^2 & -a_{13}^2 & -a_{14}^2 & \vdots & -a_{21}^2 & -a_{22}^2 & -a_{23}^2 \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} b_{11}^1 & b_{12}^1 & b_{13}^1 & b_{14}^1 & \vdots & b_{21}^1 & b_{22}^1 & b_{23}^1 \\ b_{11}^2 & b_{12}^2 & b_{13}^2 & b_{14}^2 & \vdots & b_{21}^2 & b_{22}^2 & b_{23}^2 \end{bmatrix} \mathbf{x} + \mathbf{D} \mathbf{u}$$