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線性系統
Linear Systems

Chapter 08
State Feedback & State Estimators (MIMO)

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Materials used in these lecture notes are adopted from
"Linear System Theory & Design," 3rd. Ed., by C.-T. Chen (1999)

Outline

- Introduction
- State Feedback (8.2)
- Regulation & Tracking (8.3)
- State Estimator (8.4)
- Feedback from Estimated States (8.5)
- State Feedback – Multivariable Case (8.6)
- State Estimators – Multivariable Case (8.7)
- Feedback from Estimated States – Multivariable Case (8.8)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

n-state, *p*-input, *q*-output

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}$$

$$\Rightarrow \quad \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{r}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Theorem 8.M1

The pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$, for any $p \times n$ real constant matrix $\mathbf{K}$, is controllable if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

Proof:

$$\text{rank}[\mathbf{A} - \mathbf{BK} - \lambda\mathbf{I} \;\; \mathbf{B}] = \text{rank}\left([\mathbf{A} - \lambda\mathbf{I} \;\; \mathbf{B}] \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{K} & \mathbf{I} \end{bmatrix} \right) = \text{rank}[\mathbf{A} - \lambda\mathbf{I} \;\; \mathbf{B}]$$

for all λ, including eigenvalues of **A**–**BK** and **A**

As in the SISO case,

after state feedback observability may change

Theorem 8.M3 (8.6)

Theorem 8.M3

All eigenvalues of $(\mathbf{A} - \mathbf{BK})$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant $\mathbf{K}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable.

Proof: "necessity" recall the single-input case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u$$

$$u = r - \mathbf{kx} = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - [\bar{\mathbf{k}}_1 \ \bar{\mathbf{k}}_2] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_1 & \bar{\mathbf{A}}_{12} - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_2 \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} r$$

For multi-input case only change **b** to **B** and **k** to **K**

"sufficiency" see discussions below

Cyclic Design: Theorem 8.7 (8.6.1)

Theorem 8.7

If the n-dimensional p-input pair $(\mathbf{A}, \mathbf{B})$ is controllable and if $\mathbf{A}$ is cyclic, then for almost any $p \times 1$ vector $\mathbf{v}$, the single-input pair $(\mathbf{A}, \mathbf{Bv})$ is controllable.

only one Jordan block associated with each distinct eigenvalue

⬆

A has distinct eigenvalues

- Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{Bv} = \mathbf{B}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} x \\ x \\ \alpha \\ x \\ \beta \end{bmatrix}$$

($\mathbf{A}$, $\mathbf{Bv}$) is controllable

if $\mathbf{v} \in \mathbf{R}^2$ and $\mathbf{v} \notin \{v_1+2v_2 = 0\} \cup \{v_1 = 0\}$

for a randomly selected $\mathbf{v}$,

Probability($\mathbf{v} \notin \{v_1+2v_2 = 0\} \cup \{v_1 = 0\}$) = 1.

In general, ($\mathbf{A}$, $\mathbf{Bv}$) is controllable

if $\mathbf{v} \in \mathbf{R}^p$ and $\mathbf{v} \notin \cup_i$ { hyper-plane H_i }

- Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

($\mathbf{A}$, $\mathbf{Bv}$) is not controllable for any $\mathbf{v} \in \mathbf{R}^2$,

since for this non-cyclic $\mathbf{A}$,

no controllability is possible with single input

If a $\mathbf{v} \in \mathbf{R}^p$ is chosen such that ($\mathbf{A}$, $\mathbf{Bv}$) is controllable,

then eigenvalue assignment may be accomplished

by $\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\,\mathbf{x}(t) = \mathbf{r}(t) - \mathbf{vk}\,\mathbf{x}(t)$,

where $\mathbf{k}$ is (chosen using methods for SI systems)

such that $\mathbf{A} - \mathbf{Bvk}$ has the desired eigenvalues

For non-cyclic **A**, we may resort to

Theorem 8.8

If $(\mathbf{A}, \mathbf{B})$ is controllable, then for almost any $p \times n$ real constant matrix $\mathbf{K}$, the matrix $(\mathbf{A} - \mathbf{BK})$ has only distinct eigenvalues and is, consequently, cyclic.

Proof:

(for $n = 4$) the characteristic polynomial of **A**–**BK** is

$$\Delta_f(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

where a_i's are functions of elements of **K**

A–**BK** has repeated eigenvalues

if $\Delta_f(s)$ is not coprime with

$$\Delta'_f(s) = 4s^3 + 3a_1 s^2 + 2a_2 s + a_3$$

i.e.,

$$\det \begin{bmatrix} a_4 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 2a_2 & a_4 & a_3 & 0 & 0 & 0 & 0 \\ a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 & 0 & 0 \\ a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 & a_4 & a_3 \\ 1 & 0 & a_1 & 4 & a_2 & 3a_1 & a_3 & 2a_2 \\ 0 & 0 & 1 & 0 & a_1 & 4 & a_2 & 3a_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = b(k_{ij}) = 0$$

only met by special **K**

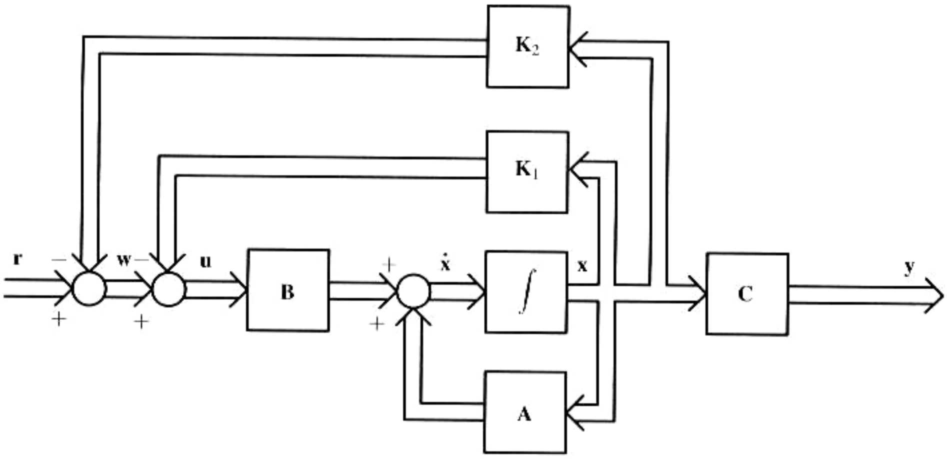
Thus for non-cyclic **A**,

may use $\mathbf{u} = \mathbf{w} - \mathbf{K}_1\mathbf{x}$ to make $\mathbf{A}-\mathbf{BK}_1$ cyclic first,

Then design $\mathbf{w} = \mathbf{r} - \mathbf{K}_2\mathbf{x} = \mathbf{r} - \mathbf{vkx}$ to assign eigenvalues, i.e.,

$$\mathbf{u} = \mathbf{w} - \mathbf{K}_1\mathbf{x} \;\Rightarrow\; \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK}_1)\mathbf{x} + \mathbf{Bw} =: \bar{\mathbf{A}}\mathbf{x} + \mathbf{Bw}$$

$$\mathbf{w} = \mathbf{r} - \mathbf{K}_2\mathbf{x} \;\Rightarrow\; \dot{\mathbf{x}} = (\bar{\mathbf{A}} - \mathbf{BK}_2)\mathbf{x} + \mathbf{Br} = (\bar{\mathbf{A}} - \mathbf{Bvk})\mathbf{x} + \mathbf{Br}$$



Combining the two feedback gains, we actually use

$$\mathbf{u} = \mathbf{r} - (\mathbf{K}_1 + \mathbf{K}_2)\mathbf{x} =: \mathbf{r} - \mathbf{K}\mathbf{x}$$

Lyapunov-Equation Method for Multi-Input State Feedback (8.6.2)

Procedure 8.M1

1. Select an $n \times n$ matrix $\mathbf{F}$ with a set of desired eigenvalues that contains no eigenvalues of $\mathbf{A}$.
2. Select an arbitrary $p \times n$ matrix $\bar{\mathbf{K}}$ such that $(\mathbf{F}, \bar{\mathbf{K}})$ is observable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{B}\bar{\mathbf{K}}$.
4. If $\mathbf{T}$ is singular, select a different $\bar{\mathbf{K}}$ and repeat the process. If $\mathbf{T}$ is nonsingular, we compute $\mathbf{K} = \bar{\mathbf{K}}\mathbf{T}^{-1}$, and $(\mathbf{A} - \mathbf{BK})$ has the set of desired eigenvalues.

   If $\mathbf{T}$ is nonsingular, the Lyapunov equation and $\mathbf{KT} = \bar{\mathbf{K}}$ imply

$$(\mathbf{A} - \mathbf{BK})\mathbf{T} = \mathbf{TF} \quad \text{or} \quad \mathbf{A} - \mathbf{BK} = \mathbf{TFT}^{-1}$$

Theorem 8.M4

If $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common, then the unique solution $\mathbf{T}$ of $\mathbf{AT}-\mathbf{TF}=\mathbf{B\bar{K}}$ is nonsingular only if $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{F}, \mathbf{\bar{K}})$ is observable.

Proof:

(**A**, **B**) not controllable

➡ $\exists\ \mathbf{p} \neq \mathbf{0}$ such that $\mathbf{p'A} = \lambda\mathbf{p'}$ and $\mathbf{p'B} = \mathbf{0}$

➡ $\mathbf{p'AT} - \mathbf{p'TF} = \lambda\mathbf{p'T} - \mathbf{p'TF} = \mathbf{p'B\bar{K}} = \mathbf{0}$

➡ $\mathbf{p'T} = \mathbf{0}$ (otherwise λ is also an eigenvalue of **F**)

➡ **T** singular

(**F**, $\mathbf{\bar{K}}$) not observable

➡ $\exists\ \mathbf{q} \neq \mathbf{0}$ such that $\mathbf{Fq} = \lambda\mathbf{q}$ and $\mathbf{\bar{K}q} = \mathbf{0}$

➡ … ➡ $\mathbf{Tq} = \mathbf{0}$ (otherwise …)

➡ **T** singular

In Procedure **8.M1**,

even with controllable (**A**, **B**) and observable (**F**, $\mathbf{\bar{K}}$),

T is may not be nonsingular,

but is nonsingular "with probability 1".

F may be chosen to have the modal form like

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

with corresponding

$$\mathbf{\bar{K}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{\bar{K}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For SISO systems the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1 \ \ \beta_2 \ \ \beta_3 \ \ \beta_4]\bar{\mathbf{x}}$$

may be utilized to facilitate state feedback gain design

Through a complex equivalence transformation

controllable MIMO state-space models also have

a "controllable canonical form" like

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & \vdots & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & \vdots & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \bar{\mathbf{x}}$$

which can be used for state feedback gain design

For example,

if the desired characteristic polynomial after state feedback is

$$\Delta_f(s) = (s^4 + \bar{\alpha}_{111}s^3 + \bar{\alpha}_{112}s^2 + \bar{\alpha}_{113}s + \bar{\alpha}_{114})(s^2 + \bar{\alpha}_{221}s + \bar{\alpha}_{222})$$

then $\bar{\mathbf{K}}$ may be chosen as

$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\alpha}_{111} - \alpha_{111} & \bar{\alpha}_{112} - \alpha_{112} & \bar{\alpha}_{113} - \alpha_{113} & \bar{\alpha}_{114} - \alpha_{114} & -\alpha_{121} & -\alpha_{122} \\ \bar{\alpha}_{211} - \alpha_{211} & \bar{\alpha}_{212} - \alpha_{212} & \bar{\alpha}_{213} - \alpha_{213} & \bar{\alpha}_{214} - \alpha_{214} & \bar{\alpha}_{221} - \alpha_{221} & \bar{\alpha}_{222} - \alpha_{222} \end{bmatrix}$$

Which obviously makes

$$\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}} = \begin{bmatrix} -\bar{\alpha}_{111} & -\bar{\alpha}_{112} & -\bar{\alpha}_{113} & -\bar{\alpha}_{114} & \vdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\bar{\alpha}_{211} & -\bar{\alpha}_{212} & -\bar{\alpha}_{213} & -\bar{\alpha}_{214} & \vdots & -\bar{\alpha}_{221} & -\bar{\alpha}_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}$$

Of course,

the transformation matrix **P** to the controllable canonical form

will be needed to find $\mathbf{K} = \bar{\mathbf{K}}\mathbf{P}$,

but the exact formula for **P** is not given here.

The system (model)

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

n-state, *p*-input, *q*-output

The estimator $\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}$

The error $\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$

satisfies $\dot{\mathbf{e}} = \underbrace{(\mathbf{A} - \mathbf{L}\mathbf{C})}\mathbf{e}$

eigenvalues can be arbitrarily assigned
by choosing **L** if (**A**, **C**) observable

Reduced-dimensional state estimator

Procedure 8.MR1

Consider the n-dimensional q-output observable pair $(\mathbf{A}, \mathbf{C})$. It is assumed that $\mathbf{C}$ has rank q.

1. Select an arbitrary $(n-q) \times (n-q)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $(n-q) \times q$ matrix $\mathbf{L}$ such that $(\mathbf{F}, \mathbf{L})$ is controllable.
3. Solve the unique $(n-q) \times n$ matrix $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$.
4. If the square matrix of order n

$$\mathbf{P} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \tag{8.77}$$

is singular, go back to Step 2 and repeat the process. If $\mathbf{P}$ is nonsingular, then the $(n-q)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y} \tag{8.78}$$

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \tag{8.79}$$

generates an estimate of $\mathbf{x}$.

Define the error signal $\mathbf{e} = \mathbf{z} - \mathbf{T}\mathbf{x}$, then

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{T}\mathbf{A}\mathbf{x} - \mathbf{T}\mathbf{B}\mathbf{u} \\ &= \mathbf{F}\mathbf{z} + (\mathbf{L}\mathbf{C} - \mathbf{T}\mathbf{A})\mathbf{x} = \mathbf{F}(\mathbf{z} - \mathbf{T}\mathbf{x}) = \mathbf{F}\mathbf{e} \end{aligned}$$

Thus $\lim_{t\to\infty} \mathbf{e}(t) = 0$ if $\mathbf{F}$ is chosen to be stable, and

$$\lim_{t\to\infty} \hat{\mathbf{x}}(t) = \lim_{t\to\infty} \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}\mathbf{x}(t) \\ \mathbf{T}\mathbf{x}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \mathbf{x}(t) = \mathbf{x}(t)$$

Theorem 8.M6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues, then the square matrix

$$\mathbf{P} := \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}$$

where $\mathbf{T}$ is the unique solution of $\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$, is nonsingular only if $(\mathbf{A}, \mathbf{C})$ is observable and $(\mathbf{F}, \mathbf{L})$ is controllable.

Proof:

(**F**, **L**) not controllable

➡ $\exists$ **p** $\neq$ **0** such that **p′F** = λ**p′** and **p′L** = **0**

➡ **p′TA** – **p′FT** = **p′TA** – λ**p′T** = **p′LC** = **0**

➡ **p′T** = **0** (otherwise λ is also an eigenvalue of **A**)

➡ **P** has L.D. rows, and is singular

(**A**, **C**) not observable

➡ $\exists$ **q** $\neq$ **0** such that **Aq** = λ**q** and **Cq** = **0**

➡ … ➡ **Tq** = **0** (otherwise ⋯)

➡ **Pq** = **0**, and **P** is singular

Controllability of (**F**, **L**) and observability of (**A**, **C**)
can not ensure the nonsingularity of **P** as in the SISO case

When full-dim. state estimator and feedback are combined,

The analysis for MIMO case is exactly

the same as that for SISO case,

Here the combination of reduced-dimensional

state estimator and feedback is discussed

The system (model)

n-state, *p*-input, *q*-output

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

The reduced-dimensional estimator

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y} \\ \hat{\mathbf{x}} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1}}_{[\mathbf{Q}_1\ \mathbf{Q}_2]} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mathbf{Q}_1\mathbf{y} + \mathbf{Q}_2\mathbf{z} \end{cases}$$

$n \times q$ $\quad n \times (n-q)$

$$[\mathbf{Q}_1\ \ \mathbf{Q}_2] \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \mathbf{Q}_1\mathbf{C} + \mathbf{Q}_2\mathbf{T} = \mathbf{I}$$

The state feedback $\quad \mathbf{u} = \mathbf{r} - \mathbf{K}\hat{\mathbf{x}} = \mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{y} - \mathbf{K}\mathbf{Q}_2\mathbf{z}$

The overall system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{C}\mathbf{x} - \mathbf{K}\mathbf{Q}_2\mathbf{z}) \\ \quad = (\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C})\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{Q}_2\mathbf{z} + \mathbf{B}\mathbf{r} \\ \dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{B}(\mathbf{r} - \mathbf{K}\mathbf{Q}_1\mathbf{C}\mathbf{x} - \mathbf{K}\mathbf{Q}_2\mathbf{z}) + \mathbf{L}\mathbf{C}\mathbf{x} \\ \quad = (\mathbf{L}\mathbf{C} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C})\mathbf{x} + (\mathbf{F} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_2)\mathbf{z} + \mathbf{T}\mathbf{B}\mathbf{r} \end{cases}$$

Matrix-vector form

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C} & -\mathbf{B}\mathbf{K}\mathbf{Q}_2 \\ \mathbf{L}\mathbf{C} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_1\mathbf{C} & \mathbf{F} - \mathbf{T}\mathbf{B}\mathbf{K}\mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{T}\mathbf{B} \end{bmatrix} \mathbf{r} \\ \mathbf{y} = [\mathbf{C}\ \ \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \end{cases}$$

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BKQ}_1\mathbf{C} & -\mathbf{BKQ}_2 \\ \mathbf{LC} - \mathbf{TBKQ}_1\mathbf{C} & \mathbf{F} - \mathbf{TBKQ}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{TB} \end{bmatrix} \mathbf{r} \\ \mathbf{y} = [\mathbf{C} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \end{cases}$$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} - \mathbf{Tx} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{T} & \mathbf{I}_{n-q} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}$$

$$\mathbf{TA} - \mathbf{FT} = \mathbf{LC}$$

$$\mathbf{Q}_1\mathbf{C} + \mathbf{Q}_2\mathbf{T} = \mathbf{I}$$

The equivalent system

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BKQ}_2 \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r} \\ \mathbf{y} = [\mathbf{C} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \end{cases}$$

separation property

The transfer matrix $\hat{\mathbf{G}}_f(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$