Fall 2007

線性系統
Linear Systems

Chapter 08
State Feedback & State Estimators (SISO)

Feng-Li Lian

NTU-EE

Sep07 – Jan08

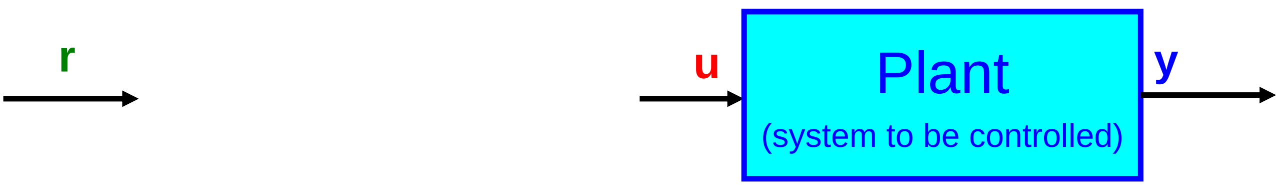
Materials used in these lecture notes are adopted from
"Linear System Theory & Design," 3rd. Ed., by C.-T. Chen (1999)

Outline

Outline

- State Feedback (8.2)
  - State Feedback in Controllable Canonical Form
  - Eigenvalue Assignment by Solving the Lyapunov Eqn
- Regulation & Tracking (8.3)
  - Robust Tracking and Disturbance Rejection
  - Stabilization
- State Estimator (8.4)
  - Full-Dimensional State Estimator
  - Reduced-Dimensional State Estimator
- Feedback from Estimated States (8.5)

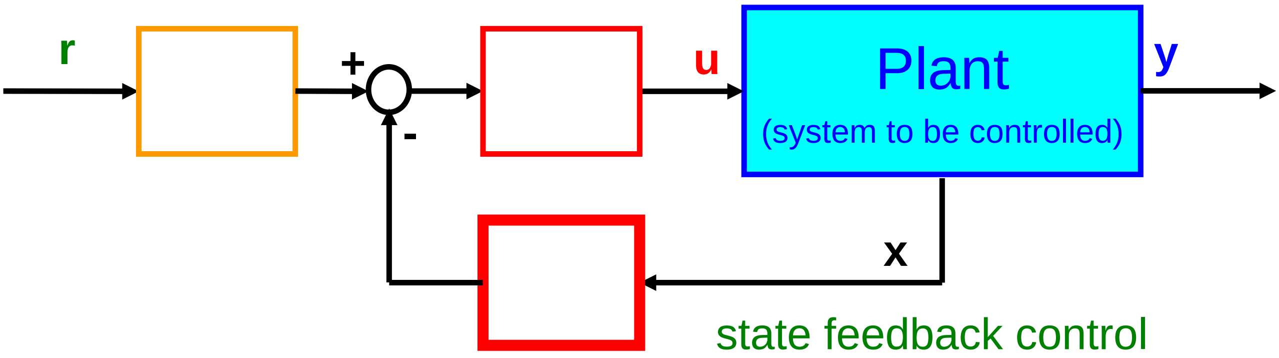
Basic Concept of Feedback Control (8.1)

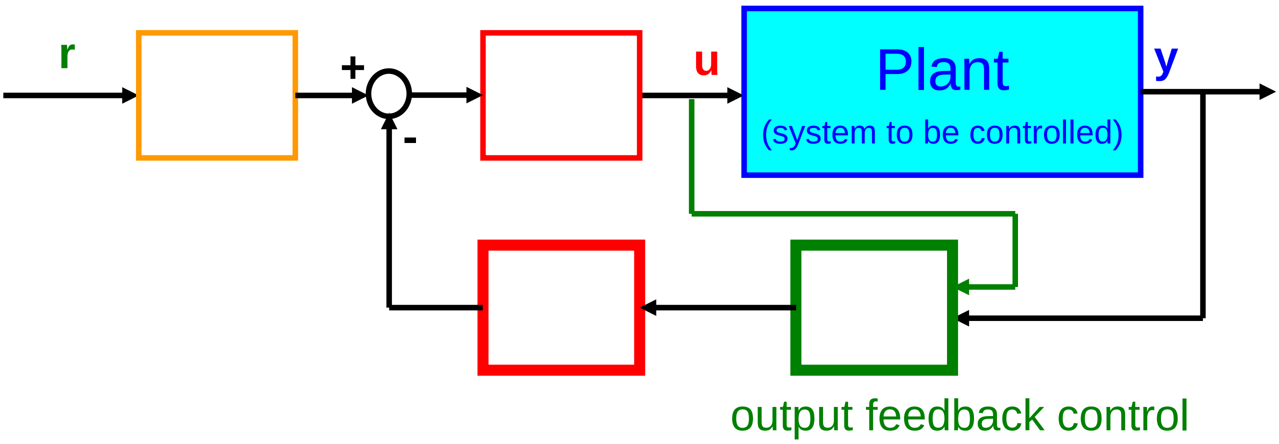


r: reference signal

Basic Concept of Feedback Control – 2

r: reference signal





State Feedback (SISO Systems) (8.2)

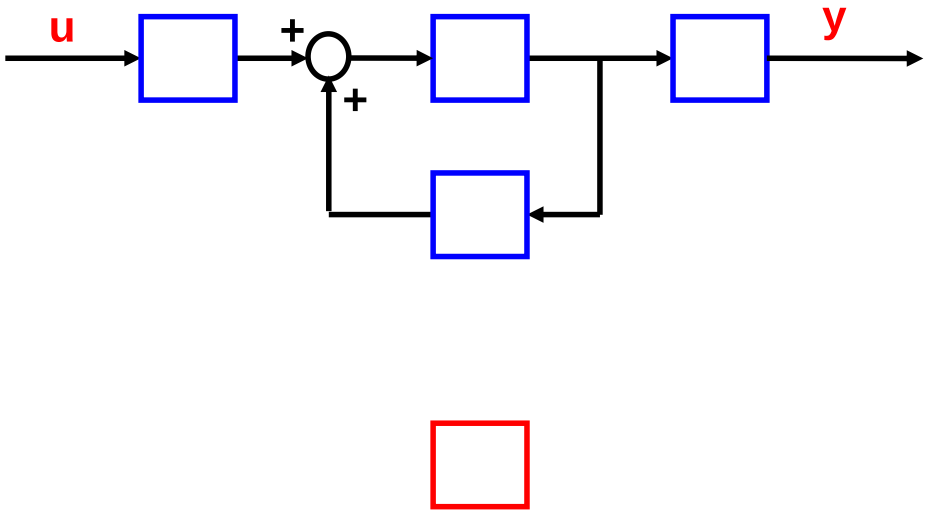
$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{c}\,\mathbf{x}(t) \end{cases}$$

$$\mathbf{u}(t) \quad =$$

- Closed-loop system :

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t) \quad = \\ \mathbf{y}(t) &= \mathbf{c}\,\mathbf{x}(t) \end{cases}$$

State Feedback (SISO Systems) – 2



Theorem 8.1 (8.2)

Theorem 8.1

The pair $(\mathbf{A} - \mathbf{bk}, \mathbf{b})$, for any $1 \times n$ real constant vector $\mathbf{k}$, is controllable if and only if $(\mathbf{A}, \mathbf{b})$ is controllable.

Proof:

Example 8.1 (8.2)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}$$

Controllability: $\mathcal{C} = \begin{bmatrix} \quad \end{bmatrix}$

Observability: $\mathcal{O} = \begin{bmatrix} \quad \end{bmatrix}$

controllable and observable

State feedback: $u = r - \begin{bmatrix} \quad \end{bmatrix} \mathbf{x}$

Feedback system:

Example 8.1 – 2

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(r - \begin{bmatrix} \quad \end{bmatrix} \mathbf{x} \right) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \quad \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ &= \begin{bmatrix} \quad \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \end{aligned}$$

Controllability: $\mathcal{C}_f = \begin{bmatrix} \quad \end{bmatrix}$

Observability: $\mathcal{O}_f = \begin{bmatrix} \quad \end{bmatrix}$

- Observability may change after state feedback

Example 8.2 (8.2)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Characteristic Polynomial:

$$\det\left(s\,\mathbf{I} - \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right) =$$

State feedback:

$$u = r - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \mathbf{x}$$

Feedback system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(r - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \mathbf{x} \right)$$

Example 8.2 – 2

$$= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -k_1 & -k_2 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$= \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

Characteristic Polynomial:

$$\det\left(s\,\mathbf{I} - \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix} \right) =$$

State Feedback in Controllable Canonical Form (8.2)

$$\{A_1, b_1, c_1, d_1\} \Longleftrightarrow \{A_2, b_2, c_2, d_2\}$$

$$x_2 = P x_1$$

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + b_1 u \\ y &= c_1 x_1 + d_1 u \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot{x}_2 &= A_2 x_2 + b_2 u \\ y &= c_2 x_2 + d_2 u \end{aligned}$$

$$\begin{aligned} A_2 &= P A_1 P^{-1} \\ b_2 &= P b_1 \\ c_2 &= c_1 P^{-1} \end{aligned}$$

State Feedback in Controllable Canonical Form – 2

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned}$$

If $\{A, b\}$ controllable

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\mathcal{C} = \left[\; b \;\; Ab \;\; A^2b \;\; A^3b \; \right]$$ is invertible

$$P_1 = \mathcal{C}^{-1} = \left[\; b \;\; Ab \;\; A^2b \;\; A^3b \; \right]^{-1}$$

$$A \left[\; b \;\; Ab \;\; A^2b \;\; A^3b \; \right] = \left[\; b \;\; Ab \;\; A^2b \;\; A^3b \; \right] \left[\quad \right]$$

$$= \left[\; Ab \;\; AAb \;\; AA^2b \;\; AA^3b \; \right] =$$

State Feedback in Controllable Canonical Form – 3

$$\mathbf{P_1}\,\mathbf{A}\,\mathbf{P_1}^{-1} = \begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} =: \mathbf{A}_1$$

$$\mathbf{b} = \left[\; \mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b} \;\right] \left[\begin{matrix} \quad \\ \quad \\ \quad \\ \quad \end{matrix}\right]$$

$$\mathbf{P_1}\,\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} =: \mathbf{b}_1$$

$$\mathbf{c}\,\mathbf{P_1}^{-1} = \mathbf{c}\left[\; \mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b} \;\right] = \left[\; \mathbf{cb} \;\; \mathbf{cAb} \;\; \mathbf{cA}^2\mathbf{b} \;\; \mathbf{cA}^3\mathbf{b} \;\right] =: \mathbf{c}_1$$

State Feedback in Controllable Canonical Form – 4

$$\mathbf{P_2} = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{P_2}\mathbf{A}_1\mathbf{P_2}^{-1}$$

$$\mathbf{b}_2 = \mathbf{P_2}\mathbf{b}_1$$

$$\mathbf{c}_1\mathbf{P_2}^{-1} = \left[\; \mathbf{cb} \;\; \mathbf{cAb} \;\; \mathbf{cA}^2\mathbf{b} \;\; \mathbf{cA}^3\mathbf{b} \;\right] \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\; (\mathbf{cb}) \;\; (\mathbf{a_1}\mathbf{cb} + \mathbf{cAb}) \;\; \cdots \;\right] \qquad = \left[\; b_1 \;\; b_2 \;\; b_3 \;\; b_4 \;\right] =: \mathbf{c}_2$$

State Feedback in Controllable Canonical Form – 5

$$\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$$

$$\mathbf{P_1} = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A^2b} & \mathbf{A^3b} \end{bmatrix}^{-1}$$

$$\Longleftrightarrow \left\{ \begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{cb} & \mathbf{cAb} & \mathbf{cA^2b} & \mathbf{cA^3b} \end{bmatrix}, d \right\}$$

$$\mathbf{P_2} = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \qquad \mathbf{P} = \mathbf{P_2 P_1}$$

$$\Longleftrightarrow \left\{ \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}, d \right\}$$

State Feedback in Controllable Canonical Form – 6

$$\dot{\mathbf{x}}_c = \mathbf{A}_c \mathbf{x}_c + \mathbf{b}_c u = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\Delta_d(s) = s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

$$\dot{\mathbf{x}}_c = \mathbf{A}_d \mathbf{x}_c + \mathbf{b}_d u = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

State Feedback in Controllable Canonical Form – 7

$$u = r - [\ k_1 \ \ k_2 \ \ k_3 \ \ k_4\ ]\mathbf{x}_c$$

$$\dot{\mathbf{x}}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} (r - [\ k_1 \ \ k_2 \ \ k_3 \ \ k_4\ ]\mathbf{x}_c)$$

$$= \left(\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -k_1 & -k_2 & -k_3 & -k_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$= \begin{bmatrix} (-a_1 - k_1) & (-a_2 - k_2) & (-a_3 - k_3) & (-a_4 - k_4) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$\dot{\mathbf{x}}_c = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

State Feedback in Controllable Canonical Form – 8

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\Delta_d(s) = s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

$$u = r - [\ k_1 \ \ k_2 \ \ k_3 \ \ k_4\ ]\mathbf{x}_c$$

$$k_i = (d_i - a_i)$$

State Feedback in Controllable Canonical Form – 9

$$\mathbf{x}_c = \mathbf{P}\mathbf{x}$$

$$\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\} \iff \left\{ \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_c, d \right\}$$

$$u = r - \mathbf{k}\mathbf{x}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\iff \left\{ \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_c - d\mathbf{k}_c, d \right\}$$

$$\mathbf{k} = \mathbf{k}_c\mathbf{P} \qquad u = r - \mathbf{k}_c\mathbf{x}_c$$

$$\Delta_d(s) = s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

In Summary – 1

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \qquad u = r - \mathbf{k}\mathbf{x}$$

$$\iff \quad \mathbf{x}_c = \mathbf{P}\mathbf{x} \qquad \mathbf{P}^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{b}_c u = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\iff \quad u = r - \mathbf{k}_c\mathbf{x}_c$$

$$\dot{\mathbf{x}}_c = \mathbf{A}_d\mathbf{x}_c + \mathbf{b}_d u = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$\Delta_d(s) = s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

$$\iff \quad u = r - \mathbf{k}_c\mathbf{x}_c = r - \mathbf{k}_c\mathbf{P}\mathbf{x} = r - \mathbf{k}\mathbf{x}$$

$$\mathbf{k} = \mathbf{k}_c\mathbf{P}$$

In Summary – 2

$\mathbf{x} \in \mathbb{R}^n$ $\qquad \mathbf{x}_c = \mathbf{P}\mathbf{x} \qquad$ $\mathbf{x}_c \in \mathbb{R}^n$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$u = r - \mathbf{k}\mathbf{x}$$

$$s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}r$$

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{b}_c u$$

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$u = r - \mathbf{k}_c\mathbf{x}_c$$

$$s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4$$

$$\dot{\mathbf{x}}_c = \mathbf{A}_d\mathbf{x}_c + \mathbf{b}_d r$$

$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{b}\mathbf{k}$ $\qquad \mathbf{k} = \mathbf{k}_c\mathbf{P} \qquad$ $\mathbf{A}_d = \mathbf{A}_c - \mathbf{b}_c\mathbf{k}_c$

Theorem 8.2 (8.2)

Theorem 8.2

Consider the state equation (A,b,c) with $n = 4$ and the characteristic polynomial

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

If (A,b) is controllable, then it can be transformed by the transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} := \mathbf{P}^{-1} = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \mathbf{A}^2\mathbf{b}\ \ \mathbf{A}^3\mathbf{b}]\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

into the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1\ \ \beta_2\ \ \beta_3\ \ \beta_4]\bar{\mathbf{x}}$$

Furthermore, the transfer function of (A,b,c) with $n = 4$ equals

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

Proof:

If with the transformation $\mathbf{Q}$, the state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is transformed into $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$, then $\hat{g}(s)$ is as shown.

$$\text{Let } \mathbf{Q}_1 = [\mathbf{b} \;\; \mathbf{A}\mathbf{b} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b}],$$

$$\mathbf{Q}_2 = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{x} = \mathbf{Q}_1 \tilde{\mathbf{x}} = \mathbf{Q}_1 \mathbf{Q}_2 \bar{\mathbf{x}} = \mathbf{Q} \bar{\mathbf{x}}$$

$$\text{Then } \tilde{\mathbf{A}} = \mathbf{Q}_1^{-1} \mathbf{A} \mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & 0 & 0 & -\alpha_3 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{bmatrix}, \qquad \tilde{\mathbf{b}} = \mathbf{Q}_1^{-1} \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{\mathbf{c}} = \mathbf{c} \mathbf{Q}_1 = [\mathbf{c}\mathbf{b} \;\; \mathbf{c}\mathbf{A}\mathbf{b} \;\; \mathbf{c}\mathbf{A}^2\mathbf{b} \;\; \mathbf{c}\mathbf{A}^3\mathbf{b}]$$

$$\text{And } \mathbf{Q}_2 \bar{\mathbf{A}} = \tilde{\mathbf{A}} \mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & -\alpha_4 \\ 1 & \alpha_1 & \alpha_2 & 0 \\ 0 & 1 & \alpha_1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{Q}_2 \bar{\mathbf{b}} = \tilde{\mathbf{b}} = [\,1 \;\; 0 \;\; 0 \;\; 0\,]'$$

$$\Rightarrow \quad \bar{\mathbf{b}} = [\,1 \;\; 0 \;\; 0 \;\; 0\,]', \qquad \bar{\mathbf{c}} = \tilde{\mathbf{c}} \mathbf{Q}_2 = [\,\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4\,]$$

Note: (1) $\beta_1 = \mathbf{c}\,\mathbf{b}$, $\beta_2 = \alpha_1 \mathbf{c}\,\mathbf{b} + \mathbf{c}\,\mathbf{A}\,\mathbf{b}$, …, but not so important here

(2) $\mathbf{Q}_1 = \mathcal{C}$, $\mathbf{Q}_2 = \bar{\mathcal{C}}^{-1}$.

Theorem 8.3 (8.2)

Theorem 8.3

If the n-dimensional state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is controllable, then by state feedback $u = r - \mathbf{k}\mathbf{x}$, where $\mathbf{k}$ is a $1 \times n$ real constant vector, the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{k}$ can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs.

Proof:

$$\bar{\mathrm{x}} = \mathrm{P}\,\mathrm{x}$$

$$\dot{\bar{\mathrm{x}}} = \bar{\mathrm{A}}\,\bar{\mathrm{x}} + \bar{\mathrm{b}}\,u \quad \Longleftrightarrow \quad \dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} + \mathrm{b}\,u$$

$$u = r - \mathrm{k}\,\mathrm{x} =$$

$$\bar{\mathrm{A}} - \bar{\mathrm{b}}\,\bar{\mathrm{k}} =$$

Theorem 8.3 – 2

Choose **P** such that

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

Use the state feedback gain

$$\bar{\mathbf{k}} = [\bar{\alpha}_1 - \alpha_1 \;\; \bar{\alpha}_2 -, \alpha_2 \;\; \bar{\alpha}_3 - \alpha_3 \;\; \bar{\alpha}_4 - \alpha_4]$$

Can obtain any desired characteristic polynomial
after state feedback

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4$$

And get

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} + \bar{\mathbf{b}}r = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

Thus $\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = \bar{\mathbf{k}}\bar{C}C^{-1}$

Double check:

$$\Delta_f(s) = \det(s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{k}) = \det\left((s\mathbf{I} - \mathbf{A})[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{k}]\right)$$

$$= \det(s\mathbf{I} - \mathbf{A})\det[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{k}]$$

$$= \Delta(s)[1 + \mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}]$$

$$\Delta_f(s) - \Delta(s) = \Delta(s)\mathbf{k}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \Delta(s)\underbrace{\bar{\mathbf{k}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}}}_{\frac{\bar{k}_1 s^3 + \bar{k}_2 s^2 + \bar{k}_3 s + \bar{k}_4}{\Delta(s)}} = \bar{k}_1 s^3 + \bar{k}_2 s^2 + \bar{k}_3 s + \bar{k}_4$$

- Zeros are not affected by state feedback
- Before state feedback: $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$

$$y = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4]\bar{\mathbf{x}}$$

$$\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

- After state feedback: $\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} + \bar{\mathbf{b}}r = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$

$$y = [\beta_1 \;\; \beta_2 \;\; \beta_3 \;\; \beta_4]\bar{\mathbf{x}}$$

$$\hat{g}_f(s) = \bar{\mathbf{c}}(s\mathbf{I} - \bar{\mathbf{A}} + \bar{\mathbf{b}}\bar{\mathbf{k}})^{-1}\bar{\mathbf{b}} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4}$$

- Observability may change with possible new pole-zero relation

Example 8.3 (8.2)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u \qquad \Delta(s) = s^2(s^2-5) = s^4 + 0\cdot s^3 - 5s^2 + 0\cdot s + 0$$

$$y = [1\ 0\ 0\ 0]\mathbf{x}$$

$$\mathbf{P}^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

If the desired eigenvalues are $-1.5\pm0.5j$ and $-1\pm j$, then

$$\Delta_f(s) = (s+1.5-0.5j)(s+1.5+0.5j)(s+1-j)(s+1+j)$$
$$= s^4 + 5s^3 + 10.5s^2 + 11s + 5$$

$$\bar{\mathbf{k}} = [5-0\ \ 10.5+5\ \ 11-0\ \ 5-0] = [5\ \ 15.5\ \ 11\ \ 5]$$

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P} = [-\tfrac{5}{3}\ \ -\tfrac{11}{3}\ \ -\tfrac{103}{12}\ \ -\tfrac{13}{3}]$$

Design Guidelines (8.2)

- Guidelines for selecting "desired eigenvalues"
  - More negative real parts of eigenvalues

    ➡ Faster response $y(t)$,

    Larger system bandwidth (noise problem), and

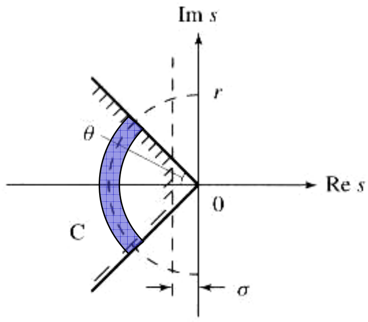
    Larger input $u(t)$ (actuator saturation problem)
  - Clustered eigenvalues

    ➡ Response $y(t)$ sensitive to parameter change, and

    Larger input $u(t)$

- Suggested methods:

(1) Let the eigenvalues uniformly spread over the crescent region

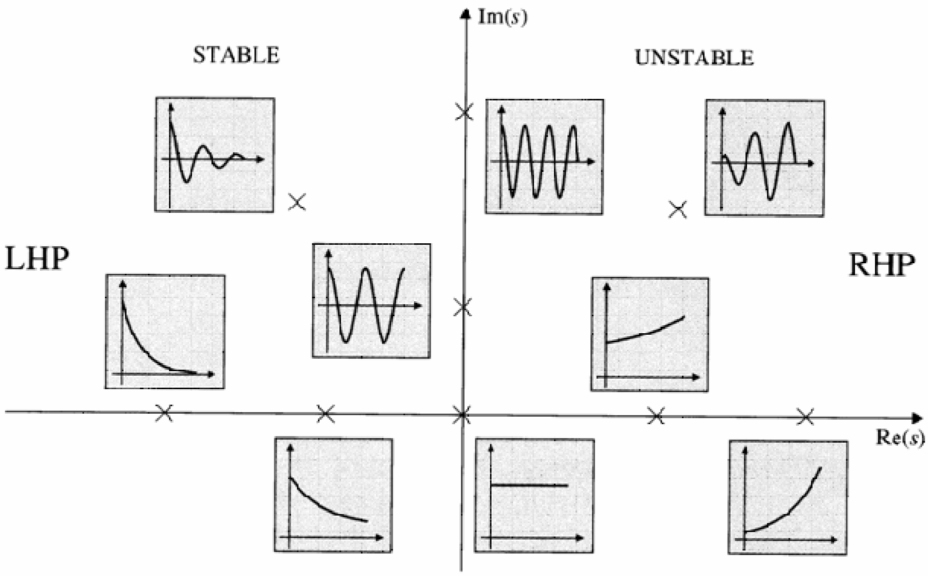


(2) Find **k** to minimize the "performance index" *J* — Optimal Control

$$J = \int_0^\infty [\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)]\, dt$$

weighting matrices

Figure 3.20
Time functions associated with points in the s-plane (LHP, left half-plane; RHP, right half-plane)



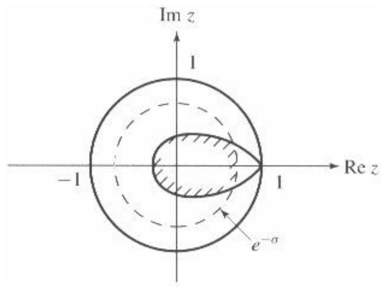
Feedback Control of Dynamic Systems, 4th Ed.,
By Franklin, Powell, Emami-Naeini, 2002

State Feedback of Discrete-Time Systems (SISO Systems) (8.2)

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k] + \mathbf{b}\,u[k]$$
$$u[k] = r[k] - \mathbf{k}\,\mathbf{x}[k]$$

$$\Rightarrow \mathbf{x}[k+1] = (\mathbf{A}-\mathbf{bk})\,\mathbf{x}[k] + \mathbf{b}\,r[k]$$

Thus everything is the same,
except that the desired eigenvalues
are different from the C.T. case.



For example,
the hatched area in the figure
is suggested.

Eigenvalue Assignment by Solving the Lyapunov Equation (8.2.1)

- A different method of computing state feedback gain for eigenvalue assignment
- Restriction: Different sets of eigenvalues wrt eig(A)

Procedure 8.1

Consider controllable $(\mathbf{A}, \mathbf{b})$, where $\mathbf{A}$ is $n \times n$ and $\mathbf{b}$ is $n \times 1$. Find a $1 \times n$ real $\mathbf{k}$ such that $(\mathbf{A} - \mathbf{bk})$ has any set of desired eigenvalues that contains no eigenvalues of $\mathbf{A}$.

1. Select an $n \times n$ matrix $\mathbf{F}$ that has the set of desired eigenvalues. The form of $\mathbf{F}$ can be chosen arbitrarily and will be discussed later.
2. Select an arbitrary $1 \times n$ vector $\bar{\mathbf{k}}$ such that $(\mathbf{F}, \bar{\mathbf{k}})$ is observable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{AT} - \mathbf{TF} = \mathbf{b}\bar{\mathbf{k}}$.
4. Compute the feedback gain $\mathbf{k} = \bar{\mathbf{k}}\mathbf{T}^{-1}$.

Note that:

In the procedure if a solution **T** exists and is nonsingular,

Then from steps **3** and **4**, we have

$$(\mathbf{A}-\mathbf{b}\mathbf{k})\mathbf{T}=\mathbf{T}\mathbf{F} \quad \text{or} \quad \mathbf{A}-\mathbf{b}\mathbf{k}=\mathbf{T}\mathbf{F}\mathbf{T}^{-1}$$

Thus **A**−**bk** and **F** have the same (assigned) eigenvalues.

A sufficient condition for **T** to exist is that

A and **F** have no common eigenvalues

(i.e., every eigenvalue must move).

Theorem 8.4

If $\mathbf{A}$ and $\mathbf{F}$ have no eigenvalues in common, then the unique solution $\mathbf{T}$ of $\mathbf{AT}-\mathbf{TF}=\mathbf{b}\bar{\mathbf{k}}$ is nonsingular if and only if $(\mathbf{A},\mathbf{b})$ is controllable and $(\mathbf{F},\bar{\mathbf{k}})$ is observable.

Proof: (for n = 4 only)

Let the characteristic polynomial of **A** be

$$\Delta(s)=s^4+\alpha_1 s^3+\alpha_2 s^2+\alpha_3 s+\alpha_4$$

Then $$\Delta(\mathbf{A})=\mathbf{A}^4+\alpha_1\mathbf{A}^3+\alpha_2\mathbf{A}^2+\alpha_3\mathbf{A}+\alpha_4\mathbf{I}=\mathbf{0}$$

And $\Delta(\bar{\lambda}_i)\neq 0$ for all eigenvalues λ_i of **F**

Thus $$\Delta(\mathbf{F}):=\mathbf{F}^4+\alpha_1\mathbf{F}^3+\alpha_2\mathbf{F}^2+\alpha_3\mathbf{F}+\alpha_4\mathbf{I}$$

has nonzero eigenvalues and is nonsingular

Also, observe the recursive relation

$$\mathbf{I}\,\mathbf{T} - \mathbf{T}\,\mathbf{I} = 0 \qquad \times$$

$$\mathbf{A}\,\mathbf{T} - \mathbf{T}\,\mathbf{F} = \mathrm{b}\,\bar{\mathrm{k}} \qquad \times$$

$$\mathbf{A}\,\mathbf{T} - \mathbf{T}\,\mathbf{F} = \mathrm{b}\,\bar{\mathrm{k}}$$

$$\mathbf{A}\,\mathbf{T} - \mathbf{T}\,\mathbf{F} = \mathrm{b}\,\bar{\mathrm{k}}$$

$$\mathbf{A}(\mathbf{AT}) - (\mathbf{TF})\mathbf{F} = \mathbf{A}^2\mathbf{T} - \mathbf{TF}^2 = \qquad \times$$

$$\mathbf{A}(\mathbf{A}^2\mathbf{T}) - (\mathbf{TF}^2)\mathbf{F} = \mathbf{A}^3\mathbf{T} - \mathbf{TF}^3 = \qquad \times$$

$$\mathbf{A}(\mathbf{A}^3\mathbf{T}) - (\mathbf{TF}^3)\mathbf{F} = \mathbf{A}^4\mathbf{T} - \mathbf{TF}^4 = \qquad (+$$

$$\Delta(\mathbf{A})\mathbf{T} - \mathbf{T}\Delta(\mathbf{F}) = \quad [\mathbf{b}\ \mathbf{Ab}\ \mathbf{A}^2\mathbf{b}\ \mathbf{A}^3\mathbf{b}] \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{k}} \\ \bar{\mathbf{k}}\mathbf{F} \\ \bar{\mathbf{k}}\mathbf{F}^2 \\ \bar{\mathbf{k}}\mathbf{F}^3 \end{bmatrix}$$

$\therefore$ **T** is

Given a set of desired eigenvalues, how to set **F** and $\bar{\mathbf{k}}$?

(1) Using the observable canonical form,

set **F** to the companion form

$$\begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix}$$

with the desired eigenvalues, and let $\bar{\mathbf{k}} = [1\ 0\ \cdots\ 0]$;

(2) Using the modal form,

set **F** to the diagonal form

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

with the desired eigenvalues, and

Let $\bar{\mathbf{k}}$ have at least one nonzero element

associated with each diagonal block of **F**,

such as $\bar{\mathbf{k}}$ = [1 1 0 1 0], [1 1 0 0 1], [1 1 1 1 1].

(the same as Example 8.3)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u$$

$$y = [1 \; 0 \; 0 \; 0]\mathbf{x}$$

The desired eigenvalues are $-1.5 \pm 0.5j$ and $-1 \pm j$, thus set

$$\mathbf{F} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1.5 & 0.5 \\ 0 & 0 & -0.5 & -1.5 \end{bmatrix}$$

No matter $\bar{\mathbf{k}}$ is set to [1 0 1 0] or [1 1 1 1],
the resulting $\mathbf{k}$ is the same as the one obtained in Example 8.3
(different $\bar{\mathbf{k}}$, different $\mathbf{T}$, but the same $\bar{\mathbf{k}}\mathbf{T}^{-1}$).

For single-input systems,

the state feedback gain $\mathbf{k}$ corresponding to

a set of pre-assigned eigenvalues is unique.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \mathbf{x} \end{cases}$$

$$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4,$$

$$u = r - \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \mathbf{x}$$

$$\left(\mathbf{A} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \right)$$

$$\boxed{s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4}$$

$$\lambda_1^c, \lambda_2^c, \lambda_3^c, \lambda_4^c,$$

State Feedback and State Estimation (MIMO) (8.6 & 8.7)

$$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \qquad \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,$$

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}\mathbf{x} \end{cases}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \mathbf{x}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}\,y$$

$$\boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4} \qquad \lambda_1^o,\ \lambda_2^o,\ \lambda_3^o,\ \lambda_4^o,$$

$$\left(\mathbf{A} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \right)$$

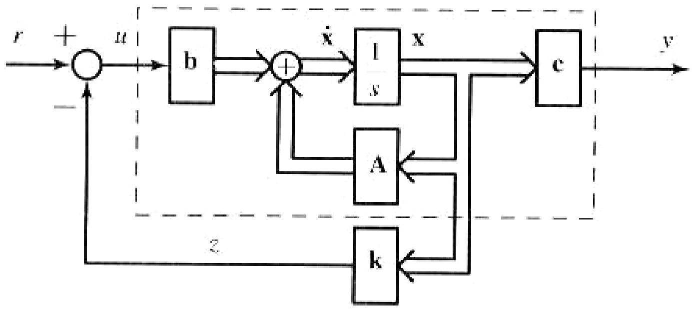
$$u = r - \mathbf{k}\,\hat{\mathbf{x}}$$

$$(\mathbf{A} - \mathbf{b}\mathbf{k})$$

$$\left(\mathbf{A} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \right)$$

$$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4} \qquad \lambda_1^c,\ \lambda_2^c,\ \lambda_3^c,\ \lambda_4^c,$$

Regulation and Tracking (8.3)



$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}\mathbf{x}(t) \end{cases}$$

Regulation:

$y(t)$ to follow $r(t) = 0$, $\forall\, t \geq 0$

Tracking:

$y(t)$ to follow $r(t) = a$, a constant, $\forall\, t \geq 0$

Servomechanism problem:

$y(t)$ to follow $r(t)$, a non-constant reference signal, $\forall\, t \geq 0$

Regulation and Tracking – 2

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}\mathbf{x}(t) \end{cases}$$

To solve the regulation problem is

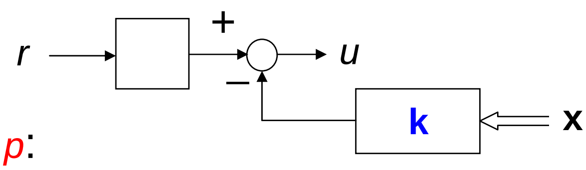
to find a state feedback gain **k**

such that **A** – **bk** is a stable matrix,

$$\begin{aligned} r(t) &= 0 \\ u(t) &= -\mathbf{k}\,\mathbf{x}(t) \\ y(t) &= \mathbf{c}\, e^{(\mathbf{A}-\mathbf{bk})t}\, \mathbf{x}(0) \end{aligned}$$

To solve the tracking problem,

we need an extra feedforward gain *p*:



$$\begin{aligned} r(t) &= a \\ u(t) &= p\, r(t) - \mathbf{k}\,\mathbf{x}(t) \\ y(t) &= \mathbf{c}\left(e^{(\mathbf{A}-\mathbf{bk})t}\,\mathbf{x}(0) + \int_0^t e^{(\mathbf{A}-\mathbf{bk})\,(t-\tau)}\, \mathbf{b}\, p\, r(\tau)\, d\tau \right) \end{aligned}$$

Regulation and Tracking – 3

With the feedforward gain and state feedback,

the transfer function is

$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = p\,\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4}$$

Thus for $y(t)$ to follow $r(t) = a$, a constant,

choose **k** such that **A** – **bk** is stable, and *p* such that

$$1 = \hat{g}_f(0) = p\frac{\beta_4}{\bar{\alpha}_4} \quad \text{or} \quad p = \frac{\bar{\alpha}_4}{\beta_4}$$

Which requires $\beta_4 \neq 0$, i.e., the plant has no zeros at $s = 0$

In practical applications, there will be disturbance, i.e., exogenous input affecting the system, and there will be uncertainty regarding the exact values of **A**, **b**, and **c**.

Robust Tracking:

Tracking with parameter uncertainty (β_4 and $\bar{\alpha}_4$ not known exactly)

Disturbance Rejection:

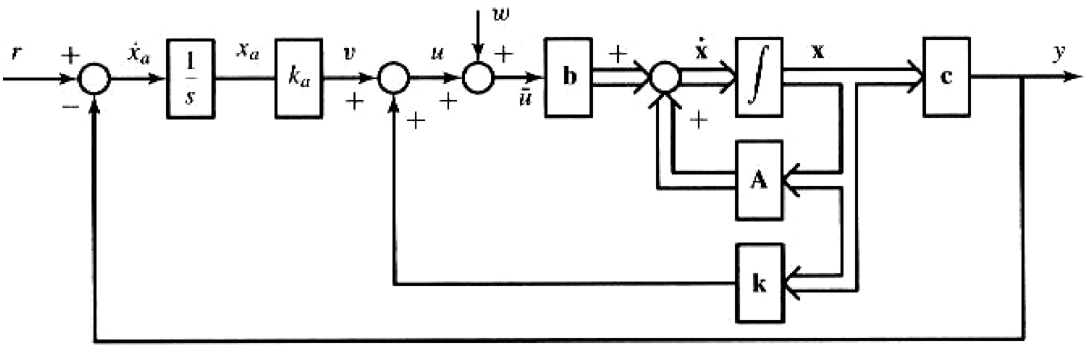
Elimination of the effect by disturbance

Plant nominal equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{b}w$$

$$y = \mathbf{c}\mathbf{x}$$

constant disturbance



An internal model control configuration with state feedback

$$\dot{x}_a = r - y = r - \mathbf{c}\mathbf{x}$$

$$u = [\mathbf{k} \;\; k_a]\begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{b}\mathbf{k} & \mathbf{b}k_a \\ -\mathbf{c} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} w$$

$$y = [\mathbf{c} \;\; 0]\begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}$$

Theorem 8.5 (8.3.1)

Theorem 8.5

If $(\mathbf{A}, \mathbf{b})$ is controllable and if $\hat{g}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ has no zero at $s = 0$, then all eigenvalues of the closed-loop system can be assigned arbitrarily by selecting a feedback gain $[\mathbf{k} \quad k_a]$.

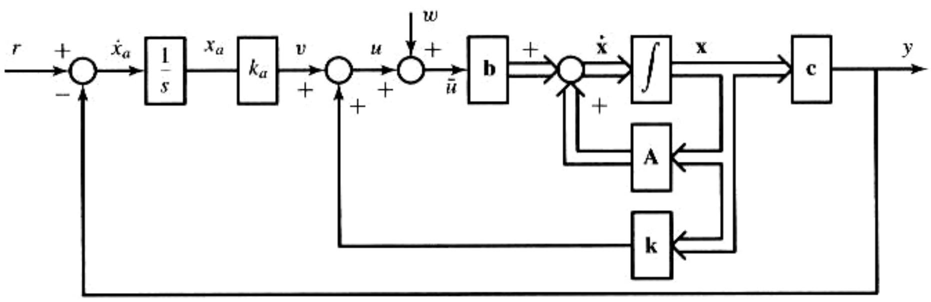
Proof:

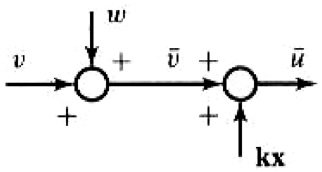
Theorem 8.5 – 2

Theorem 8.5 – 3

Theorem 8.5 – 4

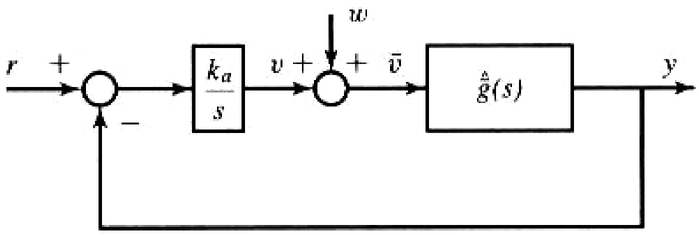
Block diagram manipulation:





$$\frac{\bar{N}(s)}{\bar{D}(s)} := \mathbf{c}(s\mathbf{I} - \mathbf{A} - \mathbf{bk})^{-1}\mathbf{b}$$

$$\bar{D}(s) = \det\,(s\mathbf{I} - \mathbf{A} - \mathbf{bk})$$



Taking determinants of the identity

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{c}(s\mathbf{I}-\mathbf{A}-\mathbf{bk})^{-1} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{I}-\mathbf{A}-\mathbf{bk} & -\mathbf{b}k_a \\ \mathbf{c} & s \end{bmatrix}$$

$$= \begin{bmatrix} s\mathbf{I}-\mathbf{A}-\mathbf{bk} & -\mathbf{b}k_a \\ 0 & s+\mathbf{c}(s\mathbf{I}-\mathbf{A}-\mathbf{bk})^{-1}\mathbf{b}k_a \end{bmatrix}$$

➡ $1 \cdot \Delta_f(s) = \bar{D}(s)\left(s + \frac{\bar{N}(s)}{\bar{D}(s)}k_a\right)$ i.e., $\Delta_f(s) = s\bar{D}(s) + k_a\bar{N}(s)$

➡ $$\hat{g}_{yw}(s) = \frac{\frac{\bar{N}(s)}{\bar{D}(s)}}{1+\frac{k_a\bar{N}(s)}{s\bar{D}(s)}} = \frac{s\bar{N}(s)}{s\bar{D}(s)+k_a\bar{N}(s)} = \frac{s\bar{N}(s)}{\Delta_f(s)}$$

For constant (step-type) disturbance $\hat{w}(s) = \bar{w}/s$

$$\hat{y}_w(s) = \frac{s\bar{N}(s)}{\Delta_f(s)}\frac{\bar{w}}{s} = \frac{\bar{w}\bar{N}(s)}{\Delta_f(s)}$$

∴ Even with (small) parameter uncertainty,

$\lim_{t\to\infty} y_w(t) = 0$,

as long as roots of $\Delta_f(s)$ all have negative real parts

➡ Robust constant-disturbance rejection

Also,

$$\hat{g}_{yr}(s) = \frac{\frac{k_a}{s}\frac{\bar{N}(s)}{\bar{D}(s)}}{1 + \frac{k_a}{s}\frac{\bar{N}(s)}{\bar{D}(s)}} = \frac{k_a\bar{N}(s)}{s\bar{D}(s) + k_a\bar{N}(s)} = \frac{k_a\bar{N}(s)}{\Delta_f(s)}$$

Thus even with parameter uncertainty

$$\hat{g}_{yr}(0) = \frac{k_a\bar{N}(0)}{0 \cdot \bar{D}(0) + k_a\bar{N}(0)} = \frac{k_a\bar{N}(0)}{k_a\bar{N}(0)} = 1$$

➡ robust tracking for constant reference

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}\mathbf{x}(t) \end{cases}$$

- Stable systems

- Stabilizable systems

- Without loss of generality, consider

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} u$$

$$u = r - \mathbf{k}\mathbf{x} = r - \bar{\mathbf{k}}\bar{\mathbf{x}} = r - [\bar{\mathbf{k}}_1 \ \bar{\mathbf{k}}_2] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_1 & \bar{\mathbf{A}}_{12} - \bar{\mathbf{b}}_c\bar{\mathbf{k}}_2 \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{b}}_c \\ \mathbf{0} \end{bmatrix} r$$

∴ Uncontrollable systems are stabilizable by state feedback
if and only if the uncontrollable part of the system is stable

$$\begin{cases} \dot{\mathrm{x}}(t) & = & \mathrm{A}\mathrm{x}(t) + \mathrm{b}u(t) \\ y(t) & = & \mathrm{c}\mathrm{x}(t) \end{cases}$$

Given **A**, **b**, **c**,

Can we estimate **x**(t)

by measuring $y(\tau)$ and $u(\tau)$ for $\tau \in [0, t]$?

State Estimator (Observer) for SISO Systems – 2

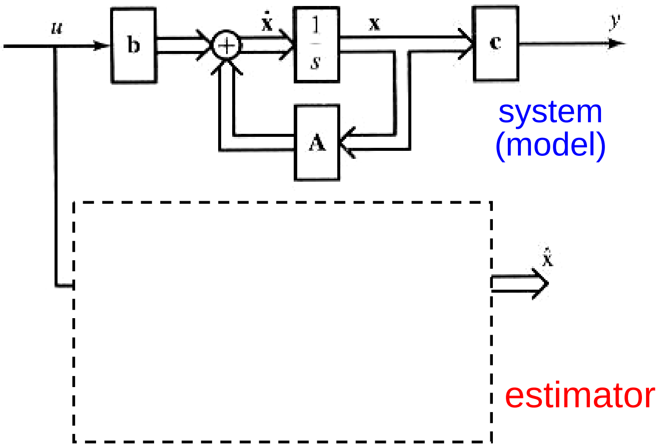
- Open-Loop Estimator:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}\mathbf{x}(t) \end{cases}$$

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t)$$

and use $y(\tau)$ and $u(\tau)$ for $\tau \in [0, t]$

to compute $\hat{\mathbf{x}}(0) = \mathbf{x}(0)$



Disadvantage 1:
Any inaccuracy in the determination of $\mathbf{x}(0)$ will make $\lim_{t\to\infty}\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\| = \infty$ for unstable $\mathbf{A}$

Disadvantage 2:
Very sensitive to parameter uncertainty in $\mathbf{A}$ and $\mathbf{b}$

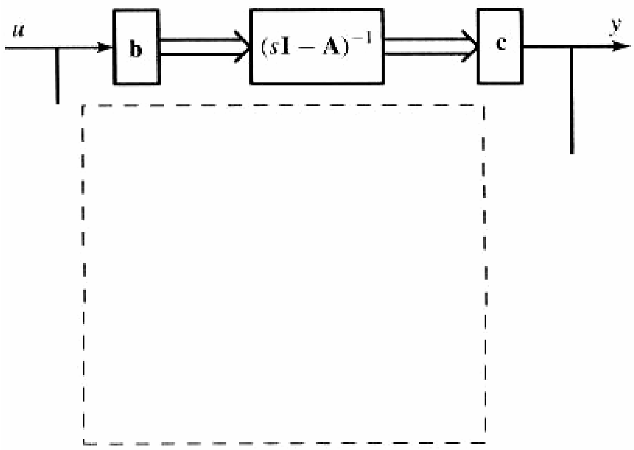
State Estimator (Observer) for SISO Systems – 3

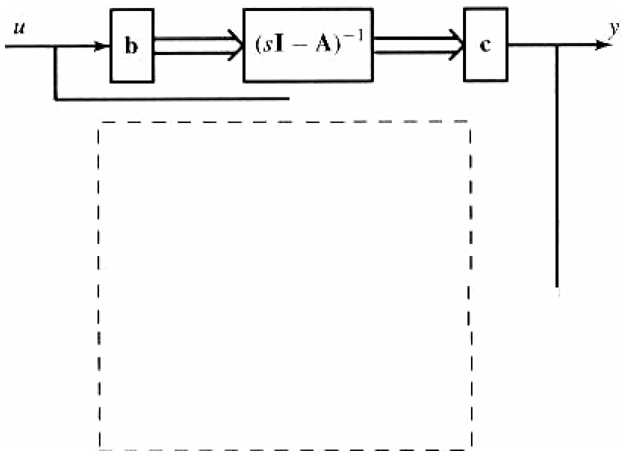
- Closed-Loop Estimator:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}\mathbf{x}(t) \end{cases}$$

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{b}u(t)$$

$$=$$





- Estimator Error:

$$e(t) := x(t) - \hat{x}(t)$$

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$

$$=$$

Theorem 8.O3

Consider the pair $(\mathbf{A}, \mathbf{c})$. All eigenvalues of $(\mathbf{A} - \mathbf{lc})$ can be assigned arbitrarily by selecting a real constant vector $\mathbf{l}$ if and only if $(\mathbf{A}, \mathbf{c})$ is observable.

Proof:

- A different method for designing state estimators:

Procedure 8.O1

1. Select an arbitrary $n \times n$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $n \times 1$ vector $\mathbf{l}$ such that $(\mathbf{F}, \mathbf{l})$ is controllable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$. This $\mathbf{T}$ is nonsingular following the dual of Theorem 8.4.
4. Then the state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y$$
$$\hat{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{z}$$

generates an estimate of $\mathbf{x}$.

$$\mathbf{e} := \mathbf{z} - \mathbf{Tx} \quad \Rightarrow \quad \begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} \\ &= \mathbf{Fz} + \mathbf{Tb}u + \mathbf{lcx} - \mathbf{TAx} - \mathbf{Tb}u \\ &= \mathbf{Fz} + \mathbf{lcx} - (\mathbf{FT} + \mathbf{lc})\mathbf{x} \\ &= \mathbf{F}(\mathbf{z} - \mathbf{Tx}) \\ &= \mathbf{Fe}\end{aligned}$$

The equation $y(t) = \mathbf{c}\,\mathbf{x}(t)$ already has one-dimensional information about $\mathbf{x}(t)$, so only an $(n-1)$-dimensional state estimator is needed to estimate the remaining information about $\mathbf{x}(t)$

Procedure 8.R1

1. Select an arbitrary $(n-1) \times (n-1)$ stable matrix $\mathbf{F}$ that has no eigenvalues in common with those of $\mathbf{A}$.
2. Select an arbitrary $(n-1) \times 1$ vector $\mathbf{l}$ such that $(\mathbf{F}, \mathbf{l})$ is controllable.
3. Solve the unique $\mathbf{T}$ in the Lyapunov equation $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$. Note that $\mathbf{T}$ is an $(n-1) \times n$ matrix.
4. Then the $(n-1)$-dimensional state equation

$$\dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Tb}u + \mathbf{l}y$$
$$\hat{\mathbf{x}} = \begin{bmatrix}\mathbf{c}\\ \mathbf{T}\end{bmatrix}^{-1}\begin{bmatrix}y\\ \mathbf{z}\end{bmatrix}$$

is an estimate of $\mathbf{x}$.

Theorem 8.6 (8.4.1)

Theorem 8.6

If $\mathbf{A}$ and $\mathbf{F}$ have no common eigenvalues, then the square matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}$$

where $\mathbf{T}$ is the unique solution of $\mathbf{TA} - \mathbf{FT} = \mathbf{lc}$, is nonsingular if and only if $(\mathbf{A}, \mathbf{c})$ is observable and $(\mathbf{F}, \mathbf{l})$ is controllable.

Theorem 8.6 – 2

Proof: (necessity for general *n*)

If ($\mathbf{F}$, $\mathbf{l}$) not controllable

If ($\mathbf{A}$, $\mathbf{c}$) not observable

Proof: (sufficiency for n = 4)

As in **Theorem 8.4**,

Let $\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$

Then

$$-\Delta(\mathbf{F})\mathbf{T} = [\mathbf{l}\ \mathbf{Fl}\ \mathbf{F}^2\mathbf{l}\ \mathbf{F}^3\mathbf{l}] \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix}$$

$$\therefore\ \mathbf{T} = -\Delta^{-1}(\mathbf{F}) C_4 \Lambda O$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ -\Delta^{-1}(\mathbf{F}) C_4 \Lambda O \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}(\mathbf{F}) \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ C_4 \Lambda O \end{bmatrix}$$

If there is a nonzero vector $\mathbf{r}$, such that $\mathbf{Pr} = \mathbf{0}$, i.e.,

$$\begin{bmatrix} \mathbf{c} \\ C_4 \Lambda O \end{bmatrix} \mathbf{r} = \begin{bmatrix} \mathbf{cr} \\ C_4 \Lambda O \mathbf{r} \end{bmatrix} = \mathbf{0}$$

Then consider

$$\mathbf{a} := \Lambda O \mathbf{r} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{cr} \\ \mathbf{cAr} \\ \mathbf{cA}^2\mathbf{r} \\ \mathbf{cA}^3\mathbf{r} \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \\ \mathbf{cr} \end{bmatrix}$$

$$C_4 \Lambda O \mathbf{r} = C_4 \mathbf{a} = C \bar{\mathbf{a}} = \mathbf{0} \ \Rightarrow\ \bar{\mathbf{a}} = \mathbf{0} \ \Rightarrow\ \mathbf{a} = \mathbf{0} \ \Rightarrow\ \mathbf{r} = \mathbf{0}.$$

C_4: $[\mathbf{l}\ \mathbf{Fl}\ \mathbf{F}^2\mathbf{l}\ \mathbf{F}^3\mathbf{l}]$

C: $[\mathbf{l}\ \mathbf{Fl}\ \mathbf{F}^2\mathbf{l}]$

($\mathbf{F}$, $\mathbf{l}$) controllable

($\mathbf{A}$, $\mathbf{c}$) observable and Λ nonsingular

In the reduced-dimensional state estimator

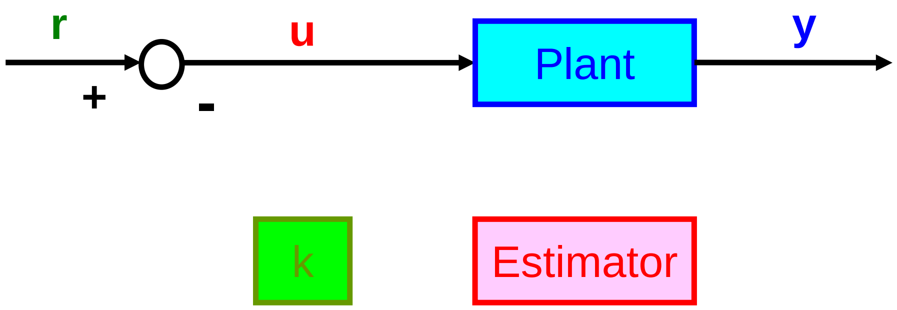
$$\begin{cases} \dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{b}u + \mathbf{l}y \\ \hat{\mathbf{x}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} y \\ \mathbf{z} \end{bmatrix} \end{cases}$$

Define the error signal $\mathbf{e} = \mathbf{z} - \mathbf{T}\mathbf{x}$, then

$$\dot{\mathbf{e}} = \dot{\mathbf{z}} - \mathbf{T}\dot{\mathbf{x}} = \mathbf{F}\mathbf{z} + \mathbf{T}\mathbf{b}u + \mathbf{l}\mathbf{c}\mathbf{x} - \mathbf{T}\mathbf{A}\mathbf{x} - \mathbf{T}\mathbf{b}u = \mathbf{F}\mathbf{e}$$

Thus $\lim_{t\to\infty} \mathbf{e}(t) = 0$ if $\mathbf{F}$ is chosen to be stable, and

$$\lim_{t\to\infty} \hat{\mathbf{x}}(t) = \lim_{t\to\infty} \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}\mathbf{x}(t) \\ \mathbf{T}\mathbf{x}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c} \\ \mathbf{T} \end{bmatrix} \mathbf{x}(t) = \mathbf{x}(t)$$



$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}\mathbf{x} \end{cases}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$

$$u = r - \mathbf{k}\hat{\mathbf{x}}$$

Feedback from Estimated States – 2

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{k}\hat{\mathbf{x}} + \mathbf{b}r \\ \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}(r - \mathbf{k}\hat{\mathbf{x}}) + \mathbf{l}\mathbf{c}\mathbf{x} \end{cases}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{b}\mathbf{k} \\ \mathbf{l}\mathbf{c} & \mathbf{A} - \mathbf{l}\mathbf{c} - \mathbf{b}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r$$

$$y = \begin{bmatrix} \mathbf{c} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Consider the equivalence transformation

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x} - \hat{\mathbf{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}}_{\mathbf{P} = \mathbf{P}^{-1}} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

Feedback from Estimated States – 3

We see

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k} & \mathbf{b}\mathbf{k} \\ \mathbf{0} & \mathbf{A} - \mathbf{l}\mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = \begin{bmatrix} \mathbf{c} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

$$\hat{g}_f(s) =$$

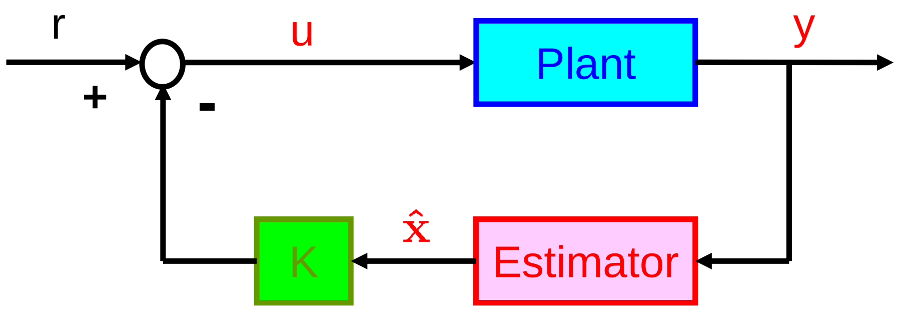
$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}\,r$$

$$y = \mathbf{c}\mathbf{x}$$

State Feedback and State Estimation (MIMO) (8.6 & 8.7) – 3

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u} \\ \mathbf{y} &= \mathbf{c}\mathbf{x} \end{cases}$$

$$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$



$$u = r - \mathbf{k}\,\hat{\mathbf{x}} \qquad \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}\,y$$

$$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4} \qquad \boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k} & \mathbf{b}\mathbf{k} \\ \mathbf{0} & \mathbf{A} - \mathbf{l}\mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$$

State Feedback and State Estimation (MIMO) (8.6 & 8.7)

$$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \qquad \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,$$

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{c}\mathbf{x} \end{cases}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \mathbf{x}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}\,y$$

$$\boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4} \qquad \lambda_1^o,\ \lambda_2^o,\ \lambda_3^o,\ \lambda_4^o,$$

$$\left(\mathbf{A} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \right)$$

$$\left(\mathbf{A} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \right)$$

$$u = r - \mathbf{k}\,\hat{\mathbf{x}}$$

$$(\mathbf{A} - \mathbf{b}\mathbf{k})$$

$$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4} \qquad \lambda_1^c,\ \lambda_2^c,\ \lambda_3^c,\ \lambda_4^c,$$

$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$ $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\,\mathbf{y}$$

$\boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4}$ $\lambda_1^o,\ \lambda_2^o,\ \lambda_3^o,\ \lambda_4^o,$

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\,\hat{\mathbf{x}}$$

$$(\mathbf{A} - \mathbf{B}\mathbf{K})$$

$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}$ $\lambda_1^c,\ \lambda_2^c,\ \lambda_3^c,\ \lambda_4^c,$

$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$ $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix} \mathbf{x}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\,\mathbf{y}$$

$\boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4}$ $\lambda_1^o,\ \lambda_2^o,\ \lambda_3^o,\ \lambda_4^o,$

$$\left(\mathbf{A} - \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \\ l_{41} & l_{42} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix} \right)$$

$$\left(\mathbf{A} - \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \end{bmatrix} \right)$$

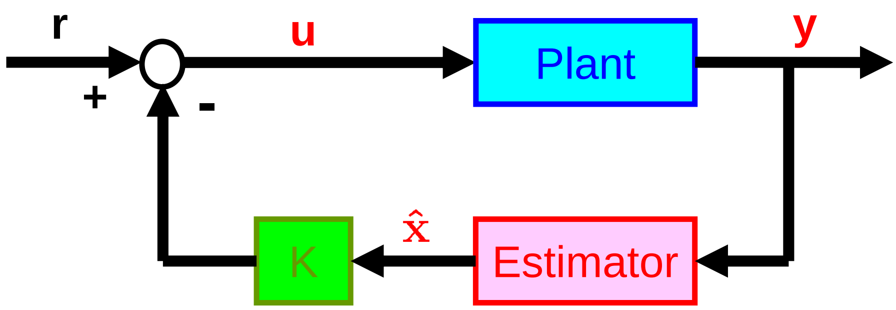
$$\mathbf{u} = \mathbf{r} - \mathbf{K}\,\hat{\mathbf{x}}$$

$$(\mathbf{A} - \mathbf{B}\mathbf{K})$$

$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}$ $\lambda_1^c,\ \lambda_2^c,\ \lambda_3^c,\ \lambda_4^c,$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

$$\boxed{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$



$$\mathbf{u} = \mathbf{r} - \mathbf{K}\,\hat{\mathbf{x}} \qquad \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\,\mathbf{y}$$

$$\boxed{s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4} \qquad \boxed{s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}$$

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$$