Solution of Homework 3, Linear Systems, Fall 2007

1 (A RLRC circuit)

The characteristic polynomial of $\mathbf{A}$ is as follows:

$$\left(-\lambda - \frac{R_1}{L}\right)\left(-\lambda - \frac{1}{CR_2}\right) + \frac{1}{LC} = 0$$

That is,

$$\lambda^2 + \lambda\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) + \left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right) = 0$$

If R_1, R_2, L, C are positive (practical assumption), the coefficients of this equation are all postive. Hence, if the roots of this equation are λ_1, λ_2, then,

$$\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) = -(\lambda_1 + \lambda_2) > 0, \qquad \left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right) = \lambda_1\,\lambda_2 > 0$$

If λ_1, λ_2 are real, then these relations hold if and only if both eigen-values are negative. The other combinations are impossible.

If λ_1, λ_2 are complex, that is, $\lambda_{1,2} = a \pm ib, \quad a, b \in R$, then $\lambda_1 + \lambda_2 = 2a$ and $\lambda_1\lambda_2 = a^2 + b^2$. Hence, $\lambda_1 + \lambda_2 < 0$ implies $a < 0$.

In summary, the system is asymptotically stable, since in both the real and complex cases the eigenvalues have negative values and negative real parts, respectively.

2 (A transistor circuit)

The characteristic polynomial of $\mathbf{A}$ is as follows:

$$\lambda^2 + \lambda\frac{h_{ie}}{L} = 0$$

The roots of this equation are $\lambda_1 = 0, \lambda_2 = -\frac{h_{ie}}{L}$.
Therefore, the system is stable, but NOT asymptotically stable.

4 (A 2-tank system)

Let $a = \frac{1}{R_1 A_1}, b = \frac{1}{R_1 A_2}, c = \frac{1}{R_2 A_2}$.

The characteristic polynomial of $\mathbf{A}$ is as follows:

$$\lambda^2 + \lambda(a + b + c) + a\,c = 0$$

Note that a, b, c are all positive, therefore

$\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 > 0$.

Hence, both roots are negative or they are complex conjugate numbers with negative real parts. In either case, the roots are in the left half of the λ-plane, and the system is asymptotically stable.

6 (A bicycle-rider system)

The characteristic polynomial of $\mathbf{A}$ is as follows:

$$\det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ -V & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & -w^2 & w^2 & -\lambda \end{bmatrix} = 0$$

The roots of this equation are $\lambda_{1,2} = 0, \lambda_{3,4} = \pm w$.
The positive eigenvalues in the right half-plane indicates instability.