# Solution of Homework 4, Linear Systems, Fall 2007

## 3 (A parallel electrical circuit)

controllability matrix:

$$\mathcal{C} = \begin{bmatrix} \frac{1}{L_1} & \frac{-1}{L_1C_1R_1} & \frac{-1}{L_1^2C_1} + \frac{1}{L_1C_1^2R_1^2} & \frac{2}{L_1^2C_1^2R_1} - \frac{1}{L_1C_1^3R_1^3} \\ \frac{1}{C_1R_1} & \frac{1}{L_1C_1} - \frac{1}{C_1^2R_1^2} & \frac{-2}{L_1C_1^2R_1} + \frac{1}{C_1^3R_1^3} & \frac{-1}{L_1^2C_1^2} + \frac{3}{L_1C_1^3R_1^2} - \frac{1}{C_1^4R_1^4} \\ \frac{1}{L_2} & \frac{-1}{L_2C_2R_2} & \frac{-1}{L_2^2C_2} + \frac{1}{L_2C_2^2R_2^2} & \frac{2}{L_2^2C_2^2R_2} - \frac{1}{L_2C_2^3R_2^3} \\ \frac{1}{C_2R_2} & \frac{1}{L_2C_2} - \frac{1}{C_2^2R_2^2} & \frac{-2}{L_2C_2^2R_2} + \frac{1}{C_2^3R_2^3} & \frac{-1}{L_2^2C_2^2} + \frac{3}{L_2C_2^3R_2^2} - \frac{1}{C_2^4R_2^4} \end{bmatrix}$$

If $L_1 = L_2, C_1 = C_2, R_1 = R2$
then rows 1 and 3 as well as rows 2 and 4 are identical.
Therefore, the system is NOT controllable.
Intuitively, the system has two parallel identical circuits. So, there is no single input that will drive one of the circuit independently.

If $c = [0, 0, 0, 1]$, the observability matrix:

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{C_2} & \frac{-1}{C_2R_2} \\ 0 & 0 & \frac{-1}{C_2^2R_2} & \frac{-1}{L_2C_2} + \frac{1}{C_2^2R_2^2} \\ 0 & 0 & a & b \end{bmatrix}$$

where $a, b$ are functions of $L_i, C_i, R_i$. Because the first two columns are zero, the system is NOT observable.

If $c = [0, 1, 0, 1]$, the observability matrix:

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{C_1} & \frac{-1}{C_1R_1} & \frac{1}{C_2} & \frac{-1}{C_2R_2} \\ \frac{-1}{C_1^2R_1} & \frac{-1}{L_1C_1} + \frac{1}{C_1^2R_1^2} & \frac{-1}{C_2^2R_2} & \frac{-1}{L_2C_2} + \frac{1}{C_2^2R_2^2} \\ \frac{-1}{L_1C_1^2} + \frac{1}{C_1^3R_1^2} & \frac{2}{L_1R_1C_1^2} - \frac{1}{C_1^3R_1^3} & \frac{-1}{L_2C_2^2} + \frac{1}{C_2^3R_2^2} & \frac{2}{L_2R_2C_2^2} - \frac{1}{C_2^3R_2^3} \end{bmatrix}$$

If $L_1 = L_2, C_1 = C_2, R_1 = R2$
then columns 1 and 3 as well as columns 2 and 4 are identical.
Therefore, the system is NOT observable.
Otherwise, the system is observable.

5 (A 3-tank system)

controllability matrix:

$$\mathcal{C} = \begin{bmatrix} b_1 & 0 & 0 & -3b_1 & 3b_2 & 0 & 15b_1 & -21b_2 & 6b_3 \\ 0 & b_2 & 0 & 2b_1 & -4b_2 & 2b_3 & -14b_1 & 28b_2 & -14b_3 \\ 0 & 0 & b_3 & 0 & 3b_2 & -3b_3 & 6b_1 & -21b_2 & 15b_3 \end{bmatrix}$$

If $b_1 = 1, b_2 = 0, b_3 = 0$ ,

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & -3 & 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{bmatrix}$$

$\text{rank}(\mathcal{C}) = 3$ , the system is controllable.

If $b_2 = 1, b_1 = 0, b_3 = 0$ ,

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 0 & 0 & -21 & 0 \\ 0 & 1 & 0 & 0 & -4 & 0 & 0 & 28 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & -21 & 0 \end{bmatrix}$$

$\text{rank}(\mathcal{C}) = 2$ , the system is NOT controllable.

If $b_3 = 1, b_1 = 0, b_2 = 0$ ,

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -14 \\ 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 15 \end{bmatrix}$$

$\text{rank}(\mathcal{C}) = 3$ , the system is controllable.

If $C = c_1 = [1, 0, 0]$, the observability matrix:

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 15 & -21 & 6 \end{bmatrix}$$

The rank is 3, the system is observable.

If $C = c_2 = [0, 1, 0]$, the observability matrix:

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -4 & 2 \\ -14 & 28 & -14 \end{bmatrix}$$

The rank $< 3$, the system is NOT observable. Because you cannot tell the difference of the level of tank-1 and tank-3 by looking at the level of tank-2.

## 6 (A bicycle-rider system)

controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B_1 V & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -B_2 V & 0 & -B_2 V^2 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & F & B_2 V w^2 & 0 & B_1 V^2 w^2 & F w^2 \\ 0 & F & B_2 V w^2 & 0 & B_1 V^2 w^2 & F w^2 & B_2 V w^4 & 0 \end{bmatrix}$$

If $F \neq 0$ , the first four columns are independent.
Therefore, the rank of $\mathcal{C}$ is 4 and the system is controllable.
However, if $F = 0$ , the rank of $\mathcal{C}$ will be 4 only if $B_2^2 w^2 \neq B_1^2 V^2$.
The condition holds if and only if $V \neq L_r w$.

observability matrix:

$$\mathcal{O} = \begin{bmatrix} -l & 1 & 0 & 0 \\ 0 & \frac{-w^2}{g} & \frac{w^2}{g} & 0 \\ -V & 0 & 0 & 0 \\ \frac{-w^2 V}{g} & 0 & 0 & \frac{w^2}{g} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-w^4}{g} & \frac{w^4}{g} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{V w^4}{g} & 0 & 0 & \frac{w^4}{g} \end{bmatrix}$$

If $V \neq 0$ , the rank of $\mathcal{O}$ is 4 and the system is observable.
However, if $V = 0$ , the rank of $\mathcal{O}$ is 3. So, the system is NOT observable. Physically, this means if the bicycle is stationary wer cannot be sure of its intial state.