Section 4.6

LTV Systems & Linearization (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov’s stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle’s theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)
U.A.S. of LTV Systems (4.6)

- Consider the linear time-varying systems:
  \[ \dot{x} = A(t)x \quad (4.29) \]

- \( x = 0 \) is an equilibrium point

- The stability behavior of the origin as an equilibrium point can be completely characterized in terms of the state transition matrix of the system.

- Form linear system theory, we know that the solution is given by
  \[ x(t) = \Phi(t, t_0) x(t_0) \]
  where \( \Phi(t, t_0) \) is the state transition matrix.

Theorem 4.11: G.U.A.S.

- Theorem 4.11

- The equilibrium point \( x = 0 \) of (4.29) is (globally) uniformly asymptotically stable \( \|X(t)\| \leq \beta (\|X(t_0)\|, t-t_0) \)

- if and only if the state transition matrix satisfies the inequality
  \[ \|\Phi(t, t_0)\| \leq k e^{-\lambda (t-t_0)}, \forall t \geq t_0 \geq 0 \]

  for some positive constants \( k \) and \( \lambda \).
Theorem 4.11: G.U.A.S.

- **Proof:**
  - Due to the linear dependence of $x(t)$ on $x(t_0)$, if the origin is U.A.S., it is globally so.

- **Sufficiency:**
  \[
  ||x(t)|| \leq ||\Phi(t, t_0)|| ||x(t_0)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}
  \]

- **Necessity:**
  Suppose the origin is U.A.S.

  Then, there is a class $\mathcal{KL}$ function $\beta$ such that
  \[
  ||x(t)|| \leq \beta(||x(t_0)||, t - t_0).
  \]

  $\forall t \geq t_0, \forall x(t_0) \in \mathbb{R}^n$

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Theorem 4.11: G.U.A.S.

- By the def. of an induced matrix norm (Appendix A), we have
  \[
  ||\Phi(t, t_0)|| = \max_{||x||=1} ||\Phi(t, t_0)x|| \leq \max_{||x||=1} \beta(||x||, t - t_0) = \beta(1, t - t_0)
  \]

- Since $\beta(1, s) \to 0$ as $s \to \infty$, there exists $T > 0$ such that $\beta(1, T) \leq 1/e$.

- For any $t \geq t_0$, let $N$ be the smallest positive integer such that $t \leq t_0 + NT$. 
Theorem 4.11: G.U.A.S.

- Divide the interval \([t_0, t_0 + (N - 1)T]\) into \((N - 1)\) equal subintervals of width \(T\) each.
- Using the transition property of \(\Phi(t, t_0)\), we can write

\[
\Phi(t, t_0) = \Phi(t, t_0 + (N - 1)T) \\
\Phi(t_0 + (N - 1)T, t_0 + (N - 2)T) \\
\vdots \\
\Phi(t_0 + T, t_0)
\]

- Hence,

\[
\|\Phi(t, t_0)\| \leq \|\Phi(t, t_0 + (N - 1)T)\| \prod_{k=1}^{k=N-1} \|\Phi(t_0 + kT, t_0 + (k - 1)T)\| \\
\leq \beta(1, 0) \prod_{k=1}^{k=N-1} \frac{1}{e} = e \beta(1, 0) e^{-N} \\
\leq e \beta(1, 0) e^{-(t-t_0)/T} = k e^{-\lambda (t-t_0)}
\]

where \(k = e \beta(1, 0)\) and \(\lambda = 1/T\). Q.E.D.

Theorem 4.11: G.U.A.S.

- Theorem 4.11 shows that, for linear systems,
  U.A.S. of the origin = E.S..

- Note that, for linear time-varying systems,
  U.A.S. cannot be characterized by the location of the eigenvalues of A.

- Thm 4.11 is not helpful as a stability test because it needs to solve the state eqn.

- However, it guarantees the existence of a Lyapunov function. See Example 4.21, for example.
Theorem 4.12: E.S.

- **Theorem 4.12**
- Let \( x = 0 \) be the E.S. E.P. of
  \[
  \dot{x} = A(t) x(t), \quad (4.29)
  \]
- Suppose \( A(t) \) is continuous and bounded.
- Let \( Q(t) \) be a cont., bdd., P.D., symm. matrix.
  \[
  0 \leq c_3 I \leq Q(t) \leq c_4 I, \quad \forall t \geq 0
  \]
- THEN, there is a cont. diff., bdd., P.D., symm. matrix \( P(t) \) that satisfies:
  \[
  -\dot{P}(t) = P(t) A(t) + A^T(t) P(t) + Q(t) \quad (4.28)
  \]
- Hence, \( V(t,x) = x^T P(t) x \) is a Lyapunov function of the system that satisfies the conditions of Thm 4.10.

**Proof:**

- Let
  \[
  P(t) = \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau
  \]
  and \( \phi(\tau; t, x) \) be the solution of (4.29) that starts at \( (t,x) \).

- Due to linearity,
  \( \phi(\tau; t, x) = \Phi(\tau, t) x. \)

- In view of the definition of \( P(t) \), we have
  \[
  x^T P(t) x = \int_t^\infty x^T \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) x d\tau
  \]
  \[
  = \int_t^\infty \phi^T(\tau; t, x) Q(\tau) \phi(\tau; t, x) d\tau
  \]
Theorem 4.12: E.S.

- Because \(|\Phi(t, t_0)| \leq k e^{-\lambda(t-t_0)}\).

- And \(0 < c_3 I \leq Q(t) \leq c_4 I\)

\[
x^T P(t) x \leq \int_t^\infty c_4 |\Phi(\tau, t)|^2 \|x\|^2 d\tau
\]

\[
\leq \int_t^\infty k^2 e^{-2\lambda(t-t)} d\tau \ c_4 \|x\|^2
\]

\[
= \frac{k^2 c_4}{2\lambda} \|x\|^2
\]

\[\triangleq c_2 \|x\|^2\]  

Exercise 3.17:

- On the other hand, since \(|A(t)|_2 \leq L, \ \forall t \geq 0\)

the solution \(\phi(\tau; t, x)\) satisfies

the lower bound

\[|\phi(\tau; t, x)|^2 \geq |x|^2 e^{-2L(\tau-t)}\]
Theorem 4.12: E.S.

- And

\[
\begin{align*}
x(\tau) &= \Phi(\tau, t)x(t) \\
\frac{d}{dt}x(\tau) &= \frac{\partial}{\partial t}\Phi(\tau, t)x(t) + \Phi(\tau, t)\frac{d}{dt}x(t) \\
0 &= \frac{\partial}{\partial t}\Phi(\tau, t)x(t) + \Phi(\tau, t)A(t)x(t) \\
\frac{\partial}{\partial t}\Phi(\tau, t) &= -\Phi(\tau, t)A(t)
\end{align*}
\]

- In particular,

\[
P(t) = \int_t^\infty \Phi^T(\tau, t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau, t) \, d\tau + \int_t^\infty \left[ \frac{\partial}{\partial t}\Phi^T(\tau, t) \right] Q(\tau)\Phi(\tau, t) \, d\tau - Q(t)
\]

\[
= -\int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t) \, d\tau A(t) - A^T(t) \int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t) \, d\tau - Q(t)
\]

\[
= -P(t)A(t) - A^T(t)P(t) - Q(t)
\]

- The fact that \(V(t, x) = x^TP(t)x\) is a Lyapunov function is shown in Ex 4.21. Q.E.D

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Linearization to Non-autonomous System

- Consider the nonlinear nonautonomous sys

\[
\dot{x} = f(t, x)
\]

where \(f : [0, \infty] \times D \rightarrow \mathbb{R}^n\) is cont. diff.

and \(D = \{ x \in \mathbb{R}^n \mid ||x||_2 < r \}\).

- Suppose the origin \(x = 0\) is an E.P.

for the systems at \(t = 0\);

that is, \(f(t, 0) = 0\) for all \(t \geq 0\).

- Furthermore, suppose

the Jacobian matrix \(\partial f/\partial x\) is bdd. and Lipschitz on \(D\), uniformly in \(t\); thus,

\[
\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\|_2 \leq L_1||x_1 - x_2||_2,
\]

\(\forall x_1, x_2 \in D, \forall t \geq 0\) for all \(1 \leq i \leq n\).
Linearization to Non-autonomous System

- By the mean value theorem,

\[ f_i(t, x) = f_i(t, 0) + \frac{\partial f_i}{\partial x}(t, z_i)x \]

where \( z_i \) is a point on the line segment connecting \( x \) to the origin.

- Since \( f(t, 0) = 0 \),
we can write \( f_i(t, x) \) as

\[ f_i(t, x) = \frac{\partial f_i}{\partial x}(t, z_i)x \]

\[ = \frac{\partial f_i}{\partial x}(t, 0)x + \left[ \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x \]

- Hence, \( f(t, x) = A(t)x + g(t, x) \)

where

\[ . \quad A(t) = \frac{\partial f}{\partial x}(t, 0) \text{ and} \]

\[ . \quad g_i(t, x) = \left[ \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x \]

Linearization to Non-autonomous System

- The function \( g(t, x) \) satisfies

\[ \|g(t, x)\|_2 \leq \left( \sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right\|_2^2 \right)^{1/2} \|x\|_2 \]

\[ \leq L \|x\|_2^2 \]

where \( L = \sqrt{n}L_1. \)

- Therefore, in a small nbhd of the origin,
we may approximate the nonlinear system by its linearization about the origin.

- The next theorem states
  Lyapunov's indirect method
  for showing E.S. of the origin
  in the nonautonomous case.
Theorem 4.13: E.S. of Non-Auto Syst.

- **Theorem 4.13:**

  - Let \( x = 0 \) be an **E.P.** for the NL sys
    \[
    \dot{x} = f(t, x)
    \]
  
  where \( f : [0, \infty) \times D \rightarrow \mathbb{R}^n \) is **cont.** diff.,
  \( D = \{ x \in \mathbb{R}^n | ||x||_2 < r \} \),
  and the **Jacobian matrix** \( \frac{\partial f}{\partial x} \) is **bdd.**
  and **Lipschitz** on \( D \), uniformly in \( t \).

  - Let
    \[
    A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}
    \]

  - Then, the origin is an **E.S.** E.P.
    for the nonlinear system
    if it is an **E.S.** E.P. for the linear system
    \[
    \dot{x} = A(t)x
    \]

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**Proof:**

- Since the linear system has an **E.S.** E.P. at
  the origin and
  \( A(t) \) is **cont.** and **bdd.**,
  **Them 4.12** ensures the existence of a
  cont. diff., bdd., P.D. symm. matrix \( P(t) \)
  that satisfies (4.28),
  where \( Q(t) \) is cont., P.D., and symm.

- We use \( V(t, x) = x^T P(t) x \) as
  a **Lyapunov func. candidate** for the NL sys.

\[
-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \quad (4.28)
\]
Theorem 4.13: E.S. of Non-Auto Syst.

- The derivative of $V(t, x)$ along the trajectories of the system is given by

$$
\dot{V}(t, x) = x^T P(t) \dot{x} + \dot{x}^T(t, x) P(t) x + x^T \dot{P}(t) x
$$

$$
= x^T P(t) f(t, x) + f^T(t, x) P(t) x + x^T \dot{P}(t) x
$$

$$
= x^T \left[ P(t) A(t) + A^T(t) P(t) + \dot{P}(t) \right] x + 2x^T P(t) g(t, x)
$$

$$
\leq -c_3 ||x||^2 + 2 c_2 L ||x||^3
$$

$$
\leq -(c_3 - 2c_2 L \rho) ||x||^2, \quad \forall ||x||_2 < \rho
$$

- Choosing $\rho < \min\{r, c_3/(2c_2 L)\}$ ensures that $\dot{V}(t, x)$ is N.D. in $||x||_2 < \rho$.

- Therefore, all the conditions of Thm 4.10 are satisfied in $||x||_2 < \rho$, and we conclude that the origin is E.S.