2.3: Qualitative Behavior Near Equilibrium Points

- Linearization, Jacobian Matrix

2.2: Multiple Equilibria

- Tunnel-diode circuit, Pendulum

2.1: Perturbed Linear Systems
2.3: Linearization at E.P. – 1

• Consider the state model:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

• \(f_1, f_2\) are continuously differentiable.

• E.P.: \(p = (p_1, p_2)\).

That is,

2.3: Linearization at E.P. – 2

• Expand the RHS into its Taylor series about \(p\):

\[
\begin{align*}
\dot{x}_1 &= \\
\dot{x}_2 &=
\end{align*}
\]
2.3: Linearization at E.P. – 3

- Let \( y_1 = x_1 - p_1, y_2 = x_2 - p_2 \)
  analyze the trajectory near \((p_1, p_2)\).

- New state equation:

\[
\dot{y}_1 = \\
\dot{y}_2 = 
\]

2.3: Linearization at E.P. – 4

- New state equation:

\[
\dot{y} = Ay
\]

where

\[
A = \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}_{x=p}
\]

\[
= \frac{\partial f}{\partial x}|_{x=p}
\]
2.3: Jacobian Matrix – 1

- \( \frac{\partial f}{\partial x} \) is called the **Jacobian matrix** of \( f(x) \)
  
  \( A \) is the **Jacobian matrix** evaluated at the **E.P.** \( p \).

- If the **origin** of the **linearized** state eqn is
  
  (1) a **stable/unstable node** with distinct eigenvalues,

  (2) a **stable/unstable focus**, or

  (3) a **saddle point**, 

2.3: Jacobian Matrix – 2

- Then in a **small neighborhood** of the **E.P.**, the trajectories of the **nonlinear** state eqn will behave like

  (1) a **stable/unstable node**, 

  (2) a **stable/unstable focus**, or

  (3) a **saddle point**.
2.1: Perturbed Linear System \( \rightarrow \) Nonlinear System – 1

- How conclusive the linearization approach is depends to a great extent on how the various qualitative phase portraits of a linear system persist under perturbations.

- For example, suppose \( A \) has distinct eigenvalues and consider \( A + \Delta A \)
  \( \Delta A: 2 \times 2 \) real matrix
  its elements have arbitrarily small magnitudes.

2.1: Perturbed Linear System \( \rightarrow \) Nonlinear System – 2

- From the perturbation theory of matrices, the eigenvalues of a matrix depend continuously on its parameters.

- That is, given an \( \epsilon > 0 \), exist a corresponding \( \delta > 0 \) the magnitude of the perturbation in each element of \( A \) is less than \( \delta \), the eigenvalues of \( (A + \Delta A) \) will lie in \( B_\epsilon \), \( B_\epsilon = \) open discs of radius \( \epsilon \) centered at the the eigenvalues of \( A \).
• Hence, after arbitrarily small perturbations, eigenvalues of \( A \) in open RHP remain in open RHP in open LHP remain in open LHP

• However, when perturbated, eigenvalues on the imaginary axis might go into either the RHP or LHP.

2.1: Perturbed Linear System \( \rightarrow \) Nonlinear System – 4

• If the EP \( x = 0 \) of \( \dot{x} = Ax \) is a node, focus, or saddle point, then the EP \( x = 0 \) of \( \dot{x} = (A + \Delta A)x \) will be of the same type for sufficiently small perturbations.

• It is quite different if the EP is a center.

• The node, focus, and saddle EPs are said to be structurally stable, while the center EP is not.
2.1: Qualitative Behavior of Nonlinear Systems

- **Nonlinear Systems:**
  \[
  \dot{x} = f(x)
  \]

- **Linear Systems:**
  \[
  \dot{x} = Ax
  \]

- In \(z\)-coordinate:
  \[
  \dot{z} = J_\gamma z
  \]

### Linearization at the E.P.

<table>
<thead>
<tr>
<th>Linearization at the E.P.</th>
<th>Change of Coordinate</th>
</tr>
</thead>
</table>
| \[
  z = M^{-1}x
  \]  |
| \[
  J_\gamma = M^{-1}AM
  \]  |

2.2: Multiple Equilibria – 1

- For **linear** systems,
  - \(\det A \neq 0\)
    - \(A\) has no zero eigenvalues,
    - \(\dot{x} = Ax\) has an isolated equilibrium point at \(x = 0\).
  - \(\det A = 0\), the system has a continuum of equilibrium points.
  - There are the **only** possible patterns.
2.2: Multiple Equilibria – 2

- For **nonlinear** systems,
  - it can have **multiple isolated** equilibrium points.

- the **tunnel-diode circuit**

- the **pendulum equation**

\[ i_R = h(v_R) \]

**Kirchhoff’s current/voltage law:**

\[ i_C + i_R - i_L = 0 \quad \text{(KCL)} \]
\[ v_C - E + Ri_L + v_L = 0 \quad \text{(KVL)} \]

**State model:**

- state: \( x_1 = v_C, x_2 = i_L \), and
- input: \( u = E \),
- \( i_C = C \frac{dv_C}{dt}, \ v_L = L \frac{di_L}{dt} \)

\[
\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2] \\
\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]
\]

---

2.2: Tunnel-Diode Circuit – 1

**Kirchhoff’s current/voltage law:**

\[ i_C + i_R - i_L = 0 \quad \text{(KCL)} \]
\[ v_C - E + Ri_L + v_L = 0 \quad \text{(KVL)} \]

**Equilibrium points:**

\[
0 = -h(x_1) + x_2 \\
0 = -x_1 - Rx_2 + u
\]

That is, the roots of:

\[
h(x_1) = \frac{E}{R} - \frac{1}{R}x_1
\]
2.2: Tunnel-Diode Circuit – 2

- **Example 2.1:**

  **State Model:**

  \[
  \dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2] \\
  \dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]
  \]

  - Assume that the circuit parameters are:
    \[ u = 1.2V, R = 1.5k\Omega, C = 2pF, L = 5\mu H \]
  - time \( t \) in nanoseconds
    \( x_2, h(x_1) \) in mA

2.2: Tunnel-Diode Circuit – 3

- **State Model:**

  \[
  \dot{x}_1 = 0.5[-h(x_1) + x_2] \\
  \dot{x}_2 = 0.2[-x_1 - 1.5x_2 + 1.2]
  \]

  and

  \[
  h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 \\
  - 226.31x_1^4 + 83.72x_1^5
  \]

- **Equilibrium Points:** (let \( \dot{x}_1 = \dot{x}_2 = 0 \))

  \[
  Q_1 = \begin{bmatrix} 0.063 \\ 0.758 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 0.285 \\ 0.61 \end{bmatrix} \quad Q_3 = \begin{bmatrix} 0.884 \\ 0.21 \end{bmatrix}
  \]
2.2: Tunnel-Diode Circuit – 4

**Example 2.3:**

The **Jacobian** matrix:

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
-0.5h'(x_1) & 0.5 \\
-0.2 & -0.3
\end{bmatrix}
\]

- Evaluated at **E.P.** \(Q_1, Q_2, Q_3\):

\[
A_1 = \begin{bmatrix}
-3.598 & 0.5 \\
-0.2 & -0.3
\end{bmatrix}, \quad (-3.57, -0.33) \quad V_1 = \begin{bmatrix}
-0.99 & -0.15 \\
-0.06 & -0.99
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
1.82 & 0.5 \\
-0.2 & -0.3
\end{bmatrix}, \quad (1.77, -0.25) \quad V_2 = \begin{bmatrix}
0.99 & -0.23 \\
-0.09 & 0.97
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
-1.427 & 0.5 \\
-0.2 & -0.3
\end{bmatrix}, \quad (-1.33, -0.4) \quad V_3 = \begin{bmatrix}
-0.98 & -0.43 \\
-0.19 & -0.89
\end{bmatrix}
\]

---

2.2: Tunnel-Diode Circuit – 5

- **\(Q_1\)** is a stable node
- **\(Q_2\)** is a saddle
- **\(Q_3\)** is a stable node
2.2: Tunnel-Diode Circuit – 6

- The two special trajectories, which approach $Q_2$, are the stable trajectories of the saddle. They form a curve that divides the plane into two halves. Which is called a separatrix.

- The separatrix partitions the plane into two regions of different qualitative behavior.

![Diagram showing the separatrix, saddle, and stable nodes]

2.2: Tunnel-Diode Circuit – 7

- In an experimental setup, we shall observe one of the two steady-state operating points $Q_1$ or $Q_3$, depending on the initial capacitor voltage and inductor current.

- The equilibrium point at $Q_2$ is never observed in practice because the ever-present physical noise would cause the trajectories to diverge from $Q_2$ even if it were possible to set up the exact initial conditions corresponding to $Q_2$. 

![Diagram showing the separatrix, saddle, and stable nodes]
2.2: Tunnel-Diode Circuit – 8

- The tunnel-diode circuit is referred as a **bistable** circuit, because it has two steady-state operating points.

- Used in **computer memory**,
  
  \[ Q_1 \rightarrow "0"
  
  \[ Q_3 \rightarrow "1"

- **Triggering** from \( Q_1 \) to \( Q_3 \) or vice versa is achieved by a **triggering** signal of **sufficiently amplitude** and **duration** that allows the trajectory to move to the **other side** of the **separatrix**.

2.2: Pendulum Equation w/ Friction – 1

Using Newton’s Second Law, write the equation of motion in the tangential direction:

\[
ml\ddot{\theta} = -mg \sin \theta - kl \dot{\theta}
\]

State model (let \( x_1 = \theta, x_2 = \dot{\theta} \)):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
\]

Equilibrium points (let \( \dot{x}_1 = \dot{x}_2 = 0 \)):

\[
\begin{align*}
0 &= x_2 \\
0 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2
\end{align*}
\]

Equilibrium points are \((n\pi, 0), n = 0, \pm 1, \pm 2, \ldots\) or, physically, \((0, 0)\) and \((\pi, 0)\).

Question? Which one is stable or unstable?
2.2: Pendulum Equation w/ Friction – 2

- **Example 2.2:**
  
  State model:
  
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -10\sin x_1 - x_2
  \end{align*}
  \]

- \((0,0)\): or \((0,0), (2\pi, 0), (-2\pi, 0)\), etc.
  a stable focus.

- \((\pi,0)\): or \((\pi, 0), (-\pi, 0)\), etc.
  a saddle.

- This picture is repeated **periodically**.
  Trajectories approach **different E.P.**
  corresponding to \# of full swings.

---

2.2: Pendulum Equation w/ Friction – 3

- **Example 2.4:**

  The Jacobian matrix:

  \[
  \frac{\partial f}{\partial x} = \begin{bmatrix}
  0 & 1 \\
  -10\cos x_1 & -1
  \end{bmatrix}
  \]

- Evaluated at E.P. \(Q_1 = (0, 0)\), \(Q_2 = (\pi, 0)\):

  \(A_1 = \begin{bmatrix}
  0 & 1 \\
  -10 & -1
  \end{bmatrix}, \quad (0.5 \pm j3.12)\)

  \(A_2 = \begin{bmatrix}
  0 & 1 \\
  10 & -1
  \end{bmatrix}, \quad (2.7, -3.7)\)

  \(V_1 = \begin{bmatrix}
  0.30 - j0.05 & 0.30 + j0.05 \\
  0.01 + j0.98 & 0.01 - j0.98
  \end{bmatrix}, \)

  \(V_2 = \begin{bmatrix}
  0.37 & -0.27 \\
  1 & 1
  \end{bmatrix}, \)
2.2: Pendulum Equation w/ Friction – 4

2.3: Qualitative Behavior Near E.P. – 1

- Phase portraits of Tunnel-Diode Circuit and Pendulum Equation show that the qualitative behavior in the vicinity of each E.P. looks just like those for linear systems.

- Tunnel-Diode circuit:
  - The trajectories near $Q_1, Q_2, Q_3$ are similar to those associated with a stable node, saddle, and stable node, respectively.
2.3: Qualitative Behavior Near E.P. – 2

- **Pendulum:**
  The trajectories near \((0, 0), (\pi, 0)\) are similar to those associated with a stable focus and saddle, respectively.

2.3: A Center

- **Example 2.5:**

  \[
  \begin{align*}
  \dot{x}_1 &= -x_2 - \mu x_1 (x_1^2 + x_2^2) \\
  \dot{x}_2 &= x_1 - \mu x_2 (x_1^2 + x_2^2)
  \end{align*}
  \]

  \[
  A = \left. \frac{\partial f}{\partial x} \right|_{0,0} = \begin{bmatrix} \end{bmatrix}
  \]

- It has an **E.P.** at the origin.
  The linearized state equation at the origin has eigenvalues \(\pm j\).
  \(\Rightarrow\) A center E.P.
The qualitative behavior of the nonlinear system can be examined by the new variables:

- a stable focus when $\mu > 0$
- an unstable focus when $\mu < 0$