2.4: Limit Cycles

2.6: Existence of Periodic Orbits

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Outline

- 2.4: Limit Cycles
  - Van der Pol Equation

- 2.6: Existence of Periodic Orbits
  - Poincare-Bendixson Criterion
  - Bendixson Criterion
  - Index Method
2.4: Limit Cycles: Harmonic Oscillator – 1

- A system oscillates
  when it has a nontrivial periodic solution:

- The image of a periodic solution
  in the phase portrait
  is a closed trajectory,
  which is usually called
  a periodic orbit or a closed orbit.

2.4: Limit Cycles: Harmonic Oscillator – 2

- In 2nd-order linear system: Oscillation
  - with eigenvalues
  - $x = 0$ is a
  - the solution:
    
    $z_1(t) =$
    
    $z_2(t) =$

    where

    - the harmonic oscillator
• Two fundamental problems
  with the linear oscillator:

  1. robustness:
     perturbation will destroy the oscillation.
     the linear oscillator is
     not structurally stable.

  2. the amplitude of oscillation is
     dependent on the initial conditions.

• It is possible to build
  physical nonlinear oscillators
  such that

  1. the nonlinear oscillator is
     structurally stable.

  2. the amplitude of oscillation
     (at steady state)
     is independent of initial conditions.
2.4: Limit Cycles: Energy Approach – 1

- The negative-resistance oscillator:
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -x_1 - \epsilon h'(x_1)x_2
  \end{align*}
  \]

  the system has only one EP at \( x_1 = x_2 = 0 \).

- The Jacobian matrix:
  \[
  A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix}
  0 & 1 \\
  -1 & -\epsilon h'(0)
  \end{bmatrix}
  \]

  \[
  \begin{align*}
  h(0) &= 0 \\
  h'(0) &< 0 \\
  h(v) &\to -\infty \text{ as } v \to -\infty
  \end{align*}
  \]

2.4: Limit Cycles: Energy Approach – 2

- Since \( h'(0) < 0 \), the origin is either an unstable node or unstable focus, depending on the value of \( \epsilon h'(0) \).

- All trajectories starting near the origin would diverge away from it and head toward infinity.

- The resistive element is "active", and supplies energy.
2.4: Limit Cycles: Energy Approach – 3

- The total energy stored in the capacitor and inductor at any time $t$ is given by:

2.4: Limit Cycles: Energy Approach – 4

- The rate of change of energy is given by:

\[ \dot{E} = \]
2.4: Limit Cycles: Energy Approach – 5

- Near the origin, the trajectory gains energy since for small $|x_1|$, $x_1h(x_1)$ is negative.

- Also, the trajectory gains energy within the strip $-a < x_1 \leq b$, and loses energy outside the strip.

- A stationary oscillation will occur if, along a trajectory, the net exchange of energy over one cycle is zero.

2.4: Limit Cycles: Van der Pol Equation

- Example 2.6 Van der Pol equation:

  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2
  \end{align*}
  \]

- For $\epsilon = 0.2, 1.0, 5.0$ are shown in Figs.
2.4: Stable & Unstable Limit Cycles

- An isolated periodic orbit is called a limit cycle.

- **stable** and **unstable** limit cycles:
  
  **stable:** \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2
  \end{align*}
  \]

  **unstable:** \[
  \begin{align*}
  \dot{x}_1 &= -x_2 \\
  \dot{x}_2 &= x_1 - \epsilon(1 - x_1^2)x_2
  \end{align*}
  \]

- Two special forms:
  - \( \epsilon \to 0 \): the averaging method
  - \( \epsilon \to \infty \): the singular perturbation method

2.6: Existence of Periodic Orbits – 1

- **Periodic orbits** in the plane are special that they divide the plane into a region **inside** the orbit and a region **outside** it.

- This makes it possible to obtain **criteria** for detecting the **presence** or **absence** of periodic orbits for **second-order systems**, which have **no generalizations** to higher order systems.
The most celebrated of these criteria are the Poincaré-Bendixson theorem, the Bendixson criterion, and the index method.

Theorem (Poincaré-Bendixson):

Let $\gamma^+$ be a bounded positive semiorbit of $\dot{x} = f(x)$, i.e., $\gamma^+(y) = \{\phi(t, y) \mid 0 \leq t < \infty\}$ and $L^+$ be its positive limit set. If $L^+$ contains no EP, then it is a periodic orbit.
2.6: Poincaré-Bendixson Theorem – 2

- **Lemma 2.1, Presence of Limit Cycles**
  (Poincaré-Bendixson Criterion):

Consider \( \dot{x} = f(x) \) and

let \( M \) be a **closed bounded** subset

of the plane,

such that

- \( M \) (1) contains **no EP**, OR
  (2) contains **only one EP**

such that the **Jacobian matrix** \( \frac{\partial f}{\partial x} \)

at this point has eigenvalues

with **positive real parts**.

(Hence, the EP is **unstable** focus or node.)

2.6: Poincaré-Bendixson Theorem – 3

- **Every trajectory** starting in \( M \)

  stays in \( M \) for all future time.

- Then, \( M \) contains a **periodic orbit** of \( \dot{x} = f(x) \).
2.6: Poincaré-Bendixson Theorem – 4

- **Intuition:**
  
  Bounded trajectories in the plane will have to approach periodic orbits or equilibrium points as time tends to infinity.

- If $M$ contains no EP, then it must contain a periodic orbit.

- If $M$ contains only one EP that satisfies the stated conditions, then in the vicinity of that point all trajectories will be moving away from it.

2.6: Poincaré-Bendixson Theorem – 5

- Therefore, we can choose a simple closed curve around the EP such that the vector field on the curve points outward.
2.6: Geometric Interpretation – 1

- Consider a simple closed curve defined by $V(x) = c$, where $V(x)$ is continuously differentiable.

- The vector field $f(x)$ at a point $x$ on the curve points inward if the inner product of $f(x)$ and the gradient vector $\nabla V(x)$ is ________; that is,

$$f(x) \cdot \nabla V(x) =$$

2.6: Geometric Interpretation – 2

- The vector field $f(x)$ points outward if $f(x) \cdot \nabla V(x)$ ________ 0.

- It is tangent to the curve if $f(x) \cdot \nabla V(x)$ ________ 0.

- Trajectories can leave a set only if the vector field points outward at some points on its boundary.
2.6: Geometric Interpretation – 3

- For a set of the form \( M = \{ V(x) \leq c \} \), for some \( c > 0 \), trajectories are trapped inside \( M \)

- For annular region of the form
  \[ M = \{ W(x) \geq c_1 \text{ and } V(x) \leq c_2 \} \]
  for some \( c_1 > 0, c_2 > 0 \)
  trajectories are trapped inside \( M \)

- The function \( f(x) \cdot \nabla V(x) \) is non-negative on \( V(x) = c_2 \)

- The function \( f(x) \cdot \nabla W(x) \) is non-negative on \( W(x) = c_1 \).

2.6: Harmonic Oscillator – 1

- **Example 2.7:**

- Consider the harmonic oscillator:

\[
\begin{align*}
\dot{x}_1 = & \quad x_2 \\
\dot{x}_2 = & \quad -x_1
\end{align*}
\]

the annual region \( M = \{ c_1 \leq V(x) \leq c_2 \} \), where \( V(x) = x_1^2 + x_2^2 \) and \( c_2 > c_1 > 0 \).

- The set \( M \) is
2.6: Harmonic Oscillator – 2

- Trajectories are trapped inside $M$ since $f(x) \cdot \nabla V(x) = 0$ everywhere.

- By PBC, there is a periodic orbit in $M$.

- PBC assures the existence of a periodic orbit, but not its uniqueness.

- Harmonic oscillator has a continuum of periodic orbits in $M$.

2.6: Example 2.8 – 1

**Example 2.8:** The system:

$$\begin{align*}
\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)
\end{align*}$$

has a unique EP at (0,0).

- the Jacobian matrix:

$$\frac{\partial f}{\partial x}\bigg|_{x=0} =$$
2.6: Example 2.8 – 2

- Let \( M = \{ V(x) \leq c \} \), where \( V(x) = x_1^2 + x_2^2 \) and \( c > 0 \).

- \( M \) is

2.6: Example 2.8 – 3

- On the surface \( V(x) = c \), we have:

\[
f(x) \cdot \nabla V(x)
\]
2.6: Negative-Resistance Oscillator – 1

- **Example 2.9:**
  The negative-resistance oscillator:

  \[ \dot{v} + \epsilon h'(v) \dot{v} + v = 0 \]

  where \( \epsilon \) is a **positive** constant

  \( h \) satisfies the conditions:

  \[ h(0) = 0, \quad h'(0) < 0, \quad \lim_{v \to \infty} h(v) = \infty, \quad \lim_{v \to -\infty} h(v) = -\infty \]

- To simplify the analysis, we impose the additional requirements:

  \[ h(0) = 0, \quad h'(0) < 0 \]

  \[ h(v) = -h(-v), \quad h(v) < 0 \text{ for } 0 < v < a, \quad h(v) > 0 \text{ for } v > a \]

2.6: Negative-Resistance Oscillator – 2

- Choose the **state variables** as:

- The **state model** as:
• A solution
  from \( A = (0, p) \) to \( E = (0, -\alpha(p)) \).

• Consider the function \( V(x) \):

• The time derivative of \( V(x) \)
• Now,

\[ \delta(p) = V(E) - V(A) = \]

• Consider the **first** term \( \delta_1(p) \):
• Consider the **second** term $\delta_2(p)$:

![Diagram with points A, B, C, D, and E connected by curves, and a graph of $-\alpha(p)$]

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• **In summary:**

![Diagram with graph of $\delta(p)$]
2.6: Negative-Resistance Oscillator – 9

- Since \( h(\cdot) \) is an odd function, due to its symmetry, if \((x_1, x_2)\) is a solution, then so is \((-x_1, -x_2)\). See Fig.

- Let \( M \) be the region enclosed by this closed curve.

- Then every trajectory starting in \( M \) at \( t = 0 \) will remain inside for all \( t \geq 0 \).

- \( M \) is closed, bounded, and has a unique EP at the origin.

2.6: Negative-Resistance Oscillator – 10

- The Jacobian matrix at the origin:
  \[
  A = \frac{\partial f}{\partial x} \bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\epsilon h'(0) \end{bmatrix}
  \]
  has eigenvalues with positive real parts since \( h'(0) < 0 \).
  By PBC, there is a closed orbit in \( M \).

- This closed orbit is unique iff \( \alpha(p) = p \).
  Only one value of \( p \), see Fig.

- Every nonequilibrium solution spirals toward the unique closed orbit.
2.6: Bendixson Criterion – 1

- To rule out the existence of periodic orbits:

- Lemma 2.2, Absence of Limit Cycles (Bendixson Criterion)
  If, on a simply connected region $D$ of the plane,
  the express $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$
  is not identially zero and does not change sign,
  then $\dot{x} = f(x)$ has no periodic orbits lying entirely in $D$.

2.6: Bendixson Criterion – 2

- Proof:
• Example 2.10: Consider the system:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) = x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) = ax_1 + bx_2 - x_1^2x_2 - x_1^3
\end{align*}
\]

and let \( D \) be the whole plane.

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• Consider \( \dot{x} = f(x) \)

• Let \( C \) be a simple closed curve not passing through any of its EP.

• Consider the orientation of the vector field \( f(x) \) at a point \( p \in C \).
• Letting \( p \) traverse \( C \) in the counterclockwise direction, the vector \( f(x) \) rotates continuously and, upon returning to the original position, must have rotated an angle \( 2\pi k \), \( k \in \mathbb{Z} \), where the angle is measured counterclockwise.

• The integer \( k \) is called the index of the closed curve \( C \).

• If \( C \) is chosen to encircle a single isolated EP \( \bar{x} \), then \( k \) is called the index of \( \bar{x} \).

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2.6: Poincaré Index of an EP – 3

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• **Lemma 2.3:**

  (a) The index of a node, a focus, or a center is \(+1\).

  (b) The index of a (hyberbolic) saddle is \(-1\).

  (c) The index of a closed orbit is \(+1\).

  (d) The index of a closed curve not encircling any EP is \(0\).

  (e) The index of a closed curve is equal to the sum of the indices of the EP within it.
• **Colollary 2.1:**

Inside any periodic orbit $\gamma$, there must be at least one EP

Suppose the EPs inside $\gamma$ are hyperbolic,

then if $N$ is the number of nodes and foci

and $S$ is the number of saddles,

it must be that $N - S = 1$.

• An EP is hyperbolic

if the Jacobian at that point has

no eigenvalues on the imaginary axis.

• If the EP is not hyperbolic,

then its index may differ from $\pm 1$.

• The index method is usually useful

in ruling out the existence of periodic orbits

in certain regions of the plane.
Example 2.11: The system:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1x_2 \\
\dot{x}_2 &= x_1 + x_2 - 2x_1x_2
\end{align*}
\]

has two EPs at (0,0) and (1,1).

The Jacobian matrices at these points are

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial x}
\end{bmatrix}
\]

\[
(0,0) = \\
(1,1) =
\]

Phase Portrait of Van der Pol Oscillator

```
Filename: expVanderPol.m
```

```
function dx = funVanderPol(t,x)
    dx = zeros(2,1);
    epsilon = 1.0;
    dx(1) = x(2);
    dx(2) = -x(1) + epsilon * (1 - x(1)^2) * x(2);
```

```
Filename: funVanderPol.m
```

```
function dx = funVanderPol(t,x)
    dx = zeros(2,1);
    epsilon = 1.0;
    dx(1) = x(2);
    dx(2) = -x(1) + epsilon * (1 - x(1)^2) * x(2);
```

```