Appendix B: Contraction Mapping

Outline

- Appendix B: Contraction Mapping
  - Vector space
  - Normed linear space
  - Banach space
  - Contraction mapping theorem
**B: Linear Vector Spaces – 1** (Appendix B; page 653)

- A **linear vector space** $\mathbf{X}$ over the field $\mathbb{R}$
  is a set of elements $x, y, z, \ldots$, called **vectors**, such that for **any two vectors** $x, y \in \mathbf{X}$

- the **sum** $x + y$ is defined, and

  - 
  - 
  - 
  - 

- and there is **zero vector** $0 \in \mathbf{X}$

  - such that

**B: Linear Vector Spaces – 2**

- For any numbers $\alpha, \beta \in \mathbb{R}$, the **scalar multiplication** $\alpha x$ is defined, and

  - 
  - 
  - 
  - 

  - and
B: Normed Linear Spaces – 1

- A **linear space** $\mathcal{X}$ is a **normed linear space** if, to each vector $x \in \mathcal{X}$, there is a **real-valued norm** $\|x\|$ that satisfies:

  - $
  \quad$

- **Convergence:**

  - Assume that $\mathcal{X}$ is a **normed linear space**.

  - A sequence $\{x_k\} \in \mathcal{X}$ **converges to** $x \in \mathcal{X}$ if

- **Closed Set:**

  - A set $S \subset \mathcal{X}$ is **closed** iff
**Cauchy Sequence:**

A sequence \( \{x_k\} \in \chi \) is said to be a Cauchy sequence if

**Banach Space:**

A normed linear space \( \chi \) is complete if

A complete normed linear space is a space.

**Theorem B.1 (Contraction Mapping):**

Let \( S \) be a closed subset of a Banach space \( \chi \) and let \( T \) be a mapping that maps \( S \) into \( S \).

Suppose that
• **THEN**
  
  − there exists a **unique** vector $x^* \in S$ satisfying
  
  − $x^*$ can be obtained by the method of **successive approximation**, starting from any **arbitrary** initial vector in $S$.

---

• **Proof:**

• Select an arbitrary $x_1 \in S$ and define the sequence $\{x_k\}$

• Since $T$ maps $S$ into $S$,

• Show that $\{x_k\}$ is Cauchy;

• Show that $x^* = T(x^*)$;

• Show that $x^*$ is the unique fixed point of $T$ in $S$. 
• Show that \( \{x_k\} \) is Cauchy.
• Show that $x^* = T(x^*)$.

• Show that $x^*$ is the unique fixed point of $T$ in $S$.

• $T$ maps $S$ into $S$.

• $T$ is a contraction mapping over $S$. 