Nonlinear Systems Analysis

Lecture 8

3.2: Dependence on Data
3.3: Sensitivity Analysis
3.4: Comparison Principle

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Outline

- 3.2: Continuous Dependence on Data
  - Gronwall-Bellman Inequality
  - Closeness of Solutions
  - Continuous Dependence on Initial State and Parameters

- 3.3: Sensitivity Analysis
  - Differentiability of Solutions
  - Sensitivity Equations

- 3.4: Comparison Principle
3.2: Continuous Dependence on Data – 1

- Here, we discuss the dependence of the solution of (3.1) on the initial state $x_0$, and the RHS function $f(t, x)$.

- Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$.

3.2: Continuous Dependence on Data – 2

- Dependence on $x_0$:
  
  - $B_\delta(y_0) = \left\{ x \in \mathbb{R}^n \mid ||x - y_0|| < \delta \right\}$

  - Given $\epsilon > 0$, there is $\delta > 0$ such that for all $z_0$ in $B_\delta(y_0)$, $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $||z(t) - y(t)|| < \epsilon$ for all $t \in [t_0, t_1]$. 
3.2: Continuous Dependence on Data – 3

- **Dependence on** \( f(t, x) \):

- One way to look at

\[
\begin{align*}
\dot{x} &= f(t, x) \\
\dot{x}_m &= f_m(t, x)
\end{align*}
\]

**IF** \( f_m(t, x) \to f(t, x) \), \( m \to \infty \)

**THEN** \( x_m \to x \)

- The other way:

- Assume that \( f \) depends **continuously** on a set of **constant parameters**;

  that is,

  \[ f = f(t, x, \lambda), \text{ where } \lambda \in \mathbb{R}^p. \]

- Let \( x(t, \lambda_0) \) be a solution of \( \dot{x} = f(t, x, \lambda_0) \)

  defined on \([t_0, t_1]\), with \( x(t_0, \lambda_0) = x_0 \).

- And \( x(t, \lambda) \) be a solution of \( \dot{x} = f(t, x, \lambda) \)

  defined on \([t_0, t_1]\), with \( x(t_0, \lambda) = x_0 \).
3.2: Continuous Dependence on Data – 3

- The solution is said to depend continuously on \( \lambda \)
  if for any \( \epsilon > 0 \), there is \( \delta > 0 \)
such that
  for all \( \lambda \) in \( B_\delta(\lambda_0) \),
  \( \dot{x} = f(t, x, \lambda) \) has a unique solution \( x(t, \lambda) \)
defined on \([t_0, t_1]\), with \( x(t_0, \lambda) = x_0 \), and
  satisfies \( ||x(t, \lambda) - x(t, \lambda_0)|| < \epsilon \)
  for all \( t \in [t_0, t_1] \).

3.2: Gronwall-Bellman Inequality – 1 (App. A, page 651)

- **Lemma A.1:** (Gronwall-Bellman Inequality)

- Let \( \lambda : [a, b] \rightarrow \mathbb{R} \) be continuous and
  \( \mu : [a, b] \rightarrow \mathbb{R} \) be cont. and nonnegative.

- **IF** a continuous function
  \( y : [a, b] \rightarrow \mathbb{R} \) satisfies
  \[
y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds
  \]
  for \( a \leq t \leq b \),

  **THEN** on the same interval
  \[
y(t) \leq \lambda(t) + \int_a^t \lambda(s) \mu(s) \exp(\int_s^t \mu(\tau) \, d\tau) \, ds
  \]
3.2: Gronwall-Bellman Inequality – 2 (App. A, page 651)

- In particular, if \( \lambda(t) \equiv \lambda \) is a constant, then

\[
y(t) \leq \lambda \exp \left[ \int_a^t \mu(\tau) \, d\tau \right]
\]

- If, in addition, \( \mu(t) \equiv \mu \geq 0 \) is a constant, then

\[
y(t) \leq \lambda \exp[\mu(t - a)]
\]

3.2: Closeness of Solutions – 1

- **Theorem 3.4:**
- Let \( f(t, x) \) be
  
  piecewise continuous in \( t \) and
  
  Lipschitz in \( x \) on \([t_0, t_1] \times W\)
  
  with a Lipschitz constant \( L \),
  
  where \( W \subset \mathbb{R}^n \) is an open connected set.

- Let \( y(t) \) and \( z(t) \) be solutions of

\[
\dot{y} = f(t, y), \quad y(t_0) = y_0
\]

and

\[
\dot{z} = f(t, z) + g(t, z), \quad z(t_0) = z_0
\]

such that \( y(t), z(t) \in W \) for all \( t \in [t_0, t_1] \).
3.2: Closeness of Solutions – 2

- Suppose that
  \[ \| g(t, x) \| \leq \mu, \quad \forall (t, x) \in [t_0, t_1] \times W \]
  for some \( \mu > 0 \).

- THEN, \( \forall t \in [t_0, t_1] \)
  \[ \| y(t) - z(t) \| \leq \| y_0 - z_0 \| \exp[L(t-t_0)] \]
  \[ + \frac{\mu}{L} \left\{ \exp[L(t-t_0)] - 1 \right\} \]

3.2: Closeness of Solutions – 3

- Proof:

- The solutions \( y(t) \) and \( z(t) \) are given by
3.2: Closeness of Solutions – 4

- Substracting the two equations and taking norms yield

3.2: Closeness of Solutions – 5

- By the **Gronwall-Bellman inequality**

\[ ||y(t) - z(t)|| \leq \]
3.2: Closeness of Solutions – 5

- Integrating the **RHS** by parts, we obtain

3.2: Dependence on Initial States & Parameters – 1

- **Theorem 3.5:**
- Let \( f(t, x, \lambda) \) be
  
  continuous in \((t, x, \lambda)\) and
  
  locally Lipschitz in \(x\) (uniformly in \(t\) and \(\lambda\))

  on \([t_0, t_1] \times D \times \{||\lambda - \lambda_0|| \leq c\}\)

  where \(D \subset \mathbb{R}^n\) is an open connected set.

- Let \(y(t, \lambda_0)\) be a solution of \(\dot{x} = f(t, x, \lambda_0)\)
  
  with \(y(t_0, \lambda_0) = y_0 \in D\).

- Suppose \(y(t, \lambda_0)\) is defined and
  
  belongs to \(D\) for all \(t \in [t_0, t_1]\).
3.2: Dependence on Initial States & Parameters – 2

- Then, given \( \epsilon > 0 \), there is \( \delta > 0 \)

  such that IF

  \( T H E N \) there is a **unique** solution \( z(t, \lambda) \)

  of \( \dot{x} = f(t, x, \lambda) \)

  defined on \( [t_0, t_1] \), with \( z(t_0, \lambda) = z_0 \),

  and \( z(t, \lambda) \) satisfies

3.2: Dependence on Initial States & Parameters – 3

- **Proof Concept:**

  ![Diagram](image-url)
3.3: Differentiability of Solutions – 1

- Suppose that
  
  \[ f(t, x, \lambda) \] is continuous in \((t, x, \lambda)\) and
  
  has continuous first partial derivatives

  w.r.t. \(x\) and \(\lambda\)

  for all \((t, x, \lambda) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p\).

- Let \(\lambda_0\) be a nominal value of \(\lambda\), and

  suppose that

  the nominal state equation \(\dot{x} = f(t, x, \lambda_0)\),

  with \(x(t_0) = x_0\)

  has a unique solution \(x(t, \lambda_0)\) over \([t_0, t_1]\).

3.3: Differentiability of Solutions – 2

- From Thm 3.5,

  for all \(\lambda\) sufficiently close to \(\lambda_0\),

  that is, \(\|\lambda - \lambda_0\|\) sufficiently small,

  \(\dot{x} = f(t, x, \lambda)\), with \(x(t_0) = x_0\)

  has a unique solution \(x(t, \lambda)\) over \([t_0, t_1]\)

  that is close to the nominal solution \(x(t, \lambda_0)\).

- The continuous differentiability of \(f\)

  w.r.t. \(x, \lambda\)

  implies the additional property that

  the solution \(x(t, \lambda)\) is differentiable

  w.r.t. \(\lambda\) near \(\lambda_0\).
• To see that, write

\[ x(t, \lambda) = \]

• Take partial derivatives wrt \( \lambda \) yields

\[ x_\lambda(t, \lambda) = \]

• Differentiating wrt \( t \),

it can be seen that \( x_\lambda(t, \lambda) \) satisfies

\[ \frac{\partial}{\partial t} x_\lambda(t, \lambda) = \]

\[ A(t, \lambda) = \]

\[ B(t, \lambda) = \]
3.3: Sensitivity Equation – 1

- For $\lambda$ sufficiently close to $\lambda_0$,
  the matrices $A(t, \lambda)$ and $B(t, \lambda)$ are defined on $[t_0, t_1]$.
  Hence, $x_\lambda(t, \lambda)$ is defined on the same interval.

- At $\lambda = \lambda_0$,
  the RHS of (3.4) depends only on the nominal solution $x(t, \lambda_0)$.

3.3: Sensitivity Equation – 2

- Let $S(t) = x_\lambda(t, \lambda_0)$;
  then $S(t)$ is the unique solution of

- $S(t)$ is called the sensitivity function, and (3.5) is called the sensitivity equation.
3.3: Sensitivity Equation – 3

- **Sensitivity functions** provide first-order estimates of the effect of parameter variations on solutions.

- For small $||\lambda - \lambda_0||$,
  
  $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$:

3.3: Sensitivity Equation – 4

- **Procedure** for calculating $S(t)$:
  
  - Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$

  - Evaluate the Jacobian matrices

    \[ A(t, \lambda_0) = \left( \frac{\partial f(t, x, \lambda)}{\partial x} \right)_{x=x(t, \lambda_0), \lambda=\lambda_0} \]

    \[ B(t, \lambda_0) = \left( \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right)_{x=x(t, \lambda_0), \lambda=\lambda_0} \]

  - Solve the sensitivity equation (3.5) for $S(t)$.
3.3: Sensitivity Equation – 5

- Alternative approach for calculating $S(t)$:

\[
\dot{x} = f(t, x, \lambda_0), \quad x(t_0) = x_0,
\]

\[
S = \left[-\frac{\partial f(t, x, \lambda)}{\partial x}\right]_{\lambda=\lambda_0} S + \left[-\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right]_{\lambda=\lambda_0} \lambda_0
\]

$S(t_0) = 0$

which is solved numerically.

3.4: Comparison Principle – 1

- Sometimes we only want to compute

the bounds of $x(t)$ without solving it.

- The **Gronwall-Bellman Inequality** is a tool.

  Another tool is the **comparison lemma**.

- Consider a differential inequality

\[
\dot{v}(t) \leq f(t, v(t))
\]

and a differential equation

\[
\dot{u}(t) = f(t, u(t)).
\]
3.4: Comparison Principle – 2

- And two facts:
  - If \( v(t) \) is differentiable at \( t \),
    
    then \( D^+ v(t) = \dot{v}(t) \).
  
  - If \( \frac{1}{h} |v(t + h) - v(t)| \leq g(t, h), \forall h \in (0, b] \)
    
    and \( \lim_{h \to 0^+} g(t, h) = g_0(t) \)
    
    then \( D^+ v(t) \leq g_0(t) \).

**upper RH derivative:**

\[
D^+ v(t) = \lim_{h \to 0^+} \sup_{h > 0} \frac{v(t + h) - v(t)}{h}
\]

The limit \( h \to 0^+ \) means that \( h \) approaches zero from above.

3.4: Comparison Principle – 3

- **Lemma 3.4:** (Comparison Lemma)

- Consider \( \dot{u} = f(t, u), \ u(t_0) = u_0 \)

  where \( f(t, u) \) is

  continuous in \( t \) and

  locally Lipschitz in \( u \),

  for all \( u \in J \subseteq \mathbb{R} \).

- Let \( [t_0, T) \) (\( T \) could be infinity)

  be the **maximal** interval of existence

  of the solution \( u(t) \),

  and suppose \( u(t) \in J \) for all \( t \in [t_0, T) \).
3.4: Comparison Principle – 3

- Let \( v(t) \) be a continuous function whose upper RH derivative \( D^+v(t) \) satisfies the differential inequality
  \[ D^+v(t) \leq f(t, v(t)), \ v(t_0) \leq u_0 \]
  with \( v(t) \in J \) for all \( t \in [t_0, T) \).

- Then, \( v(t) \leq u(t) \) for all \( t \in [t_0, T) \).

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3.4: Example 3.8 – 1

- Example 3.8:

- Consider the scalar D.E.
  \[ \dot{x} - f(x) - (1 + x^2)x, \ x(0) = a \]

  has a unique solution on \([0, t_1)\),
  for some \( t_1 > 0 \),
  because \( f(x) \) is local Lipschitz.
3.4: Example 3.8 – 2

- \( v(t) \) is differentiable and its derivative is given by

- Hence,
  
  \( v(t) \) satisfies the differential inequality

3.4: Example 3.8 – 3

- Let \( u(t) \) be the solution of the D.E.

- Then, by the comparison lemma,
  
  the solution \( x(t) \) is defined for all \( t \geq 0 \)

  and satisfies