Comparison Functions
(Lyapunov Stability)

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Outline

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- Autonomous Systems (4.1 L9)
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- Input-to-State Stability (4.9, L17)
Comparison Functions

- From autonomous to non-autonomous

- The sol of the nonautonomous syst
  \[ \dot{x} = f(t, x) \]
  starting at \( x(t_0) = x_0 \),
  depends on both \( t \) and \( t_0 \).

- Should refine the definitions
  to let stability hold
  uniformly in the initial time \( t_0 \).

- Need some special comparison functions.

**Definition 4.2: Class K Functions**

- \( \alpha : [0, a) \to [0, \infty) \) is a continuous function.

- **IF** \( \alpha(\cdot) \) is strictly increasing and \( \alpha(0) = 0 \),
  **THEN** it is said to belong to class \( \mathcal{K} \)

- **IF** \( a = \infty \) and \( \alpha(r) \to \infty \) as \( r \to \infty \),
  **THEN** it is said to belong to class \( \mathcal{K}_\infty \)
Definition 4.3: Class KL Functions

- $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is a **continuous** function.

- **IF**
  1. For each fixed $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ w.r.t. $r$ and,
  2. For each fixed $r$, the mapping $\beta(r, s)$ is decreasing w.r.t. $s$ and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$

- **THEN** it is said to belong to class $\mathcal{KL}$

Example 4.16

- $\alpha(r) = \tan^{-1}(r)$

- $\alpha(r) = r^c, \quad c > 0$

- $\alpha(r) = \min\{r, r^2\}$
Example 4.16

- $\beta(r, s) = re^{-s}$, $c > 0$

- $\beta(r, s) = r/(ksr + 1)$, $k > 0$

Lemma 4.2

- $\alpha_1$ and $\alpha_2$ be class $\mathcal{K}$ functions on $[0, a)$,
  $\alpha_3$ and $\alpha_4$ be class $\mathcal{K}_\infty$ functions
  and $\beta$ be a class $\mathcal{KL}$ function.

- Denote the inverse of $\alpha_i$ by $\alpha_i^{-1}$.

- THEN
  - $\alpha_1^{-1}$ is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$
  - $\alpha_3^{-1}$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.
  - $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$
  - $\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$
  - $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$. 
Lemma 4.3

- Let $V : D \to R$ be a continuous P.D. function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin.

- Let $B_r \subset D$ for some $r > 0$.

- Then, there exist class $\mathcal{K}$ functions $\alpha_1, \alpha_2$, defined on $[0, r]$, such that

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$$

for all $x \in B_r$.

Lemma 4.3

- If $D = \mathbb{R}^n$,

  $\alpha_1, \alpha_2$ will be defined on $[0, \infty)$ and the foregoing inequality will hold $\forall x \in \mathbb{R}^n$.

- Moreover, if $V(x)$ is radially unbounded, then $\alpha_1, \alpha_2$ can be chosen to belong to class $\mathcal{K}_\infty$.

- If $V(x) = x^T P x$,

  $$\lambda_{\min}(P)||x||_2^2 \leq x^T P x \leq \lambda_{\max}(P)||x||_2^2$$
Lemma 4.4

- Consider the scalar autonomous D.E.
  \[ \dot{y} = -\alpha(y), \quad y(t_0) = y_0 \]
  where \(\alpha(\cdot)\) is a local Lipschitz class \(K\) function defined on \([0, a)\).

- For all \(0 \leq y_0 < a\),
  it has a unique solution \(y(t) \forall t \geq t_0\).

- Moreover, \(y(t) = \sigma(y_0, t - t_0)\)
  where \(\sigma\) is a class \(KL\) function defined on \([0, a) \times [0, \infty)\).

Lemma 4.4: Examples

- If \(\dot{y} = -ky\), \(k > 0\),

- If \(\dot{y} = -ky^2\), \(k > 0\),
Comparison Funs & Lyapunov Analysis

- For the proof of Thm 4.1:
  - Want to choose $\beta, \delta$
    such that $B_\delta \subset \Omega_\beta \subset B_r$
  - So, for a P.D. function $V(x)$,
  - Because

- Then

Also, want to show that

when $\dot{V}(x)$ is N.D., $x(t) \to 0$ as $t \to \infty$.

- Using Lemma 4.3,
  - there is a class $\mathcal{K}$ function $\alpha_3$
    such that $\dot{V}(x) \leq -\alpha_3(||x||)$
  - Hence, $\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$

- Comparison lemma (Lemma 3.4) shows that
  - $V(x(t))$ is bounded by the solution of
    $\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))$
Comparison Funs & Lyapunov Analysis

- **Lemma 4.2** shows that
  \( \alpha_3 \circ \alpha_2^{-1} \) is a class \( \mathcal{K} \) function.

- **Lemma 4.4** shows that
  the solution is \( y(t) = \beta(y(0), t) \),
  where \( \beta \) is a class \( \mathcal{KL} \) function.

- Consequently, \( V(x(t)) \) satisfies
  \[ V(x(t)) \leq \beta(V(x(0)), t), \]
  which shows that \( V(x(t)) \to 0 \) as \( t \to \infty \).

Estimates of \( x(t) \)

- \( V(x(t)) \leq V(x(0)) \) implies
  \[ \alpha_1(||x(t)||) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(||x(0)||) \]

- Hence, \( ||x(t)|| \leq \alpha_1^{-1}(\alpha_2(||x(0)||)) \),
  where \( \alpha_1^{-1} \circ \alpha_2 \) is a class \( \mathcal{K} \) function.

- Similarly, \( V(x(t)) \leq \beta(V(x(0)), t) \) implies
  \[ \alpha_1(||x(t)||) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(||x(0)||), t) \]

- Therefore, \( ||x(t)|| \leq \alpha_1^{-1}(\beta(\alpha_2(||x(0)||), t))) \),
  where \( \alpha_1^{-1}(\beta(\alpha_2(r), t)) \) is a class \( \mathcal{KL} \) func.