Lecture Note 14

Section 4.6

LTV Systems & Linearization (Lyapunov Stability)

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Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov’s stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle’s theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)
• Consider the linear time-varying systems:

\[ \dot{x} = A(t)x \]  \hspace{1cm} (4.29)

• \( x = 0 \) is an equilibrium point

• The stability behavior of the origin as an equilibrium point can be completely characterized in terms of the state transition matrix of the system.

• Form linear system theory, we know that the solution is given by

\[ x(t) = \Phi(t, t_0) x(t_0) \]

where \( \Phi(t, t_0) \) is the state transition matrix.
Theorem 4.11: G.U.A.S.

- The equilibrium point $x = 0$ of (4.29) is (globally) uniformly asymptotically stable

- if and only if the state transition matrix satisfies the inequality

$$||\Phi(t, t_0)|| \leq k e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

for some positive constants $k$ and $\lambda$. 

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Theorem 4.11: G.U.A.S.: Proof
Theorem 4.11: G.U.A.S.

- **Theorem 4.11** shows that,
  
  for linear systems,
  
  U.A.S. of the origin = E.S.
  
- Note that,
  
  for linear time-varying systems,
  
  U.A.S. cannot be characterized
  
  by the location of the eigenvalues of A.
  
- **Thm 4.11** is **not helpful** as a stability test
  
  because it needs to **solve the state eqn**.
  
- However, it guarantees
  
  the existence of a Lyapunov function.
  
  See Example 4.21, for example.

Theorem 4.12: E.S.

- Let \( x = 0 \) be the E.S. E.P. of
  
  \[
  \dot{x} = A(t) \ x(t), \quad (4.29)
  \]

- Suppose \( A(t) \) is **continuous** and **bounded**.

- Let \( Q(t) \) be a cont., bdd., P.D., symm.
  
  matrix.

  \[
  0 \leq c_3 I \leq Q(t) \leq c_4 I, \quad \forall t \geq 0
  \]
Theorem 4.12: E.S.

- **THEN**, there is a **cont. diff., bdd., P.D., symm.** matrix $P(t)$ satisfying

\[ -\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \quad (4.28) \]

- Hence, $V(t, x) = x^T P(t) x$ is a **Lyapunov function** of the system that satisfies the conditions of **Thm 4.10**.
Linearization to Non-autonomous System

- Consider the nonlinear nonautonomous sys

\[ \dot{x} = f(t, x) \]

where \( f : [0, \infty] \times D \rightarrow \mathbb{R}^n \) is cont. diff.
and \( D = \{ x \in \mathbb{R}^n \mid \|x\|_2 < r \} \).

- Suppose the origin \( x = 0 \) is an E.P.
  for the systems at \( t = 0 \);
  that is, \( f(t, 0) = 0 \) for all \( t \geq 0 \).
Linearization to Non-autonomous System

- Furthermore, suppose the Jacobian matrix \([\partial f/\partial x]\) is bdd. and Lipschitz on \(D\), uniformly in \(t\); thus,

\[
\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\|_2 \leq L_1 \|x_1 - x_2\|_2,
\]

\(\forall x_1, x_2 \in D, \forall t \geq 0\) for all \(1 \leq i \leq n\).

- By the mean value theorem,

\[
f_i(t, x) = f_i(t, 0) + \frac{\partial f_i}{\partial x}(t, z_i)x
\]

where \(z_i\) is a point on the line segment connecting \(x\) to the origin.

- Since \(f(t, 0) = 0\), we can write \(f_i(t, x)\) as

\[
f_i(t, x) = \frac{\partial f_i}{\partial x}(t, z_i)x
\]

\[
= \frac{\partial f_i}{\partial x}(t, 0)x + \left[ \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x
\]

- Hence, \(f(t, x) = A(t)x + g(t, x)\)

where

\[
A(t) = \frac{\partial f}{\partial x}(t, 0) \text{ and } \]

\[
g_i(t, x) = \left[ \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x
\]
The function $g(t, x)$ satisfies
\[
\|g(t, x)\|_2 \leq \left( \sum_{i=1}^{n} \left\| \frac{\partial f_i(t, z_i)}{\partial x} - \frac{\partial f_i(t, 0)}{\partial x} \right\|_2^2 \right)^{1/2} \|x\|_2 \\
\leq L \|x\|_2^2
\]
where $L = \sqrt{n}L_1$.

Therefore, in a small nbhd of the origin, we may approximate the nonlinear system by its linearization about the origin.

The next theorem states Lyapunov's indirect method for showing E.S. of the origin in the nonautonomous case.
Theorem 4.13: E.S. of Non-Auto Syst.

- Let \( x = 0 \) be an E.P. for the NL sys

\[
\dot{x} = f(t, x)
\]

where \( f : [0, \infty) \times D \rightarrow \mathbb{R}^n \) is cont. diff.,

\( D = \{ x \in \mathbb{R}^n \mid \|x\|_2 < r \} \),

and the Jacobian matrix \( [\partial f/\partial x] \) is bdd.

and Lipschitz on \( D \), uniformly in \( t \).

- Let

\[
A(t) = \frac{\partial f}{\partial x}(t, x) \bigg|_{x=0}
\]

Theorem 4.13: E.S. of Non-Auto Syst.

- Then, the origin is an E.S. E.P.

for the nonlinear system

if it is an E.S. E.P. for the linear system

\[
\dot{x} = A(t)x
\]