Outline

- Representation of a CT Signal by Its Samples: The Sampling Theorem
- Reconstruction of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals

The Sampling Theorem

- Representation of CT Signals by its Samples

\[ x_1(kT) = x_2(kT) = x_3(kT) \]
The Sampling Theorem

- **The Sampling Theorem:**
  
  \( x(t) \) : a band-limited signal with \( X(jw) = 0 \) for \( |w| > w_M \)

  if \( w_s > 2w_M \) where \( w_s = \frac{2\pi}{T} \)

  \[ x(t) \] is uniquely determined by \( x(nT), n = 0, \pm 1, \pm 2, \ldots \),

  \[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

  \[ x_p(t) = x(t)p(t) \]

  \[ x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \]

  \( 2w_M \) : Nyquist rate

  \( w_M \) : Nyquist frequency

- **Impulse-Train Sampling:**

  \( p(t) \) : sampling function

  \( T \) : sampling period

  \( w_s = \frac{2\pi}{T} \) : sampling frequency

  \[ x(t) \]

  \[ p(t) \]

  \[ x_p(t) \]

  \[ x(t) \]

  \( x_p(t) = x(t)p(t) \)

  \[ x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \]

- **Exact Recovery by an Ideal Lowpass Filter:**

  \( x(t) \)

  \( X(jw) \)

  \( P(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - kw_s) \)

  From multiplication property,

  \[ X_p(jw) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta)P(j(w - \theta))d\theta \]

  \[ = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(w - kw_s)) \]

  \( w_s > 2w_M \)

  \( w_s < 2w_M \)
The Sampling Theorem

- **Sampling with Zero-Order Hold:**

  ![Diagram of Sampling with Zero-Order Hold]

  - $x(t)$ input
  - $x_0(nT)$ output
  - $x_p(t)$ reconstructed signal

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Reconstruction of a Signal from its Samples Using Interpolation

- **Exact Interpolation:**

  - $x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$
  - $h(t) = \frac{w_cT \sin(w_c t)}{\pi w_c t}$
  - $x_r(t) = x_p(t) * h(t)$
  - $x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT)$
  - $x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_cT \sin(w_c(t-nT))}{\pi w_c(t-nT)}$

  ![Diagram of Exact Interpolation]
Reconstruction of a Signal from its Samples Using Interpolation

Higher-Order Holds:

\[ x(t) \rightarrow x_s(t) \rightarrow x_d(t) \]

(a) \hspace{2cm} (b)

\[ h(t) \]

(c)

\[ H_1(jw) = \frac{1}{T} \left( \frac{\sin(wT/2)}{w/2} \right)^2 \]

(d)

\[ H_0(jw) \]

Effect of Under-sampling: Aliasing

Overlapping in Frequency-Domain: Aliasing

\[ x(t) = \cos(w_0t) \]

Effect of Under-sampling: Aliasing

Overlapping in Frequency-Domain: Aliasing

\[ w_s > 2w_0 \]

\[ w_s < 2w_0 \]

\[ w_s = 2w_0 \]

\[ w_s \neq 2w_0 \]
Effect of Under-sampling: Aliasing

- **Overlapping in Frequency-Domain: Aliasing**

\[ w_0 = \frac{w_s}{6} \]

\[ w_0 = \frac{2w_s}{6} \]

\[ w_0 = \frac{4w_s}{6} \]

\[ w_0 = \frac{5w_s}{6} \]

- **Strobe Effect:**

\[ w_0 = 100 \text{ rad/sec} \]

\[ w = \pm w_s \pm w_0 \]

\[ = \pm 20, -20 \]

\[ w_s = 120 \text{ rad/sec} \]

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**Discrete-Time Processing of Continuous-Time Signals**

- **Discrete-Time Processing of CT Signals:**

  ![Diagram](image1.png)

- **C/D or A-to-D (ADC) and D/C or D-to-A (DAC):**

  ![Diagram](image2.png)

  - **C/D**: continuous-to-discrete-time conversion
  - **A-to-D**: analog-to-digital converter
  - **D/C**: discrete-to-continuous-time conversion
  - **D-to-A**: digital-to-analog converter

- **C/D Conversion:**

  ![Diagram](image3.png)

  **$X_c(j\omega)$**

  **$X_p(j\omega)$**

  **$X_d(\Omega)$**
**C/D Conversion:**

\[ x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT) \]

\[ x_p(jw) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-jwnT} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(w - kw_0)) \]

\[ x_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\Omega n} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)) \]

**Frequency-Domain Illustration:**

\[ Y_d(e^{j\Omega}) = X_d(e^{j\Omega}) \]

\[ Y_p(jw) = H_p(jw) X_p(jw) \]

\[ Y_c(jw) = H_c(jw) X_c(jw) \]
### Discrete-Time Processing of Continuous-Time Signals

**Frequency-Domain Illustration:**

\[
X_C(j\omega) \quad Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})
\]

\[
X_p(j\omega) \quad Y_p(j\omega) = H_p(j\omega)X_p(j\omega)
\]

\[
X_d(e^{j\Omega}) \quad Y_c(j\omega) = H_c(j\omega)X_c(j\omega)
\]

---

**CT & DT Frequency Responses:**

\[
Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T})
\]

\[
H_c(j\omega) = \begin{cases} 
H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\
0, & |\omega| > \omega_s/2 
\end{cases}
\]

---

**In Summary**

- **Sampling & Reconstruction (7.3)**

**The Sampling Theorem:**

- If the **sampling instants** are sufficiently close, very little is lost by sampling a CT signal.

- If the **sampling points** are too far apart, much of the information about a signal can be lost.

- So, when a CT signal can be uniquely given by its sampled version?
Theorem 7.1: (Shannon, 1949)

- $f(t)$: a continuous-time signal
- $F(w)$: the Fourier transform of $f(t)$
  \[ F(w) = 0 \text{ outside } (-w_0, w_0) \]
- $w_s$: sampling frequency

$\Rightarrow$ If $w_s > 2w_0$

Then $f(t)$ can be computed by:

\[
f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t - kh)/2)}{w_s(t - kh)/2} \sin \frac{w_s(t - kh)}{2}
\]

Reconstruction:

- $F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t) dt$
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} F(w) dw$
- $F_s(w) = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(w + kw_s)$
  \[
  = \sum_{k=-\infty}^{\infty} C_k e^{-i kw_s} \quad C_k = \frac{1}{w_s} \int_{0}^{w_s} e^{i kw_s} F_s(w) dw
  \]
  \[
  = \sum_{k=-\infty}^{\infty} f(kh)e^{-i kw_s}
  \]
- $F(w) = \begin{cases} hF_s(w) & |w| \leq \frac{w_s}{2} \\ 0 & |w| > \frac{w_s}{2} \end{cases}$

Shannon Reconstruction:

- For periodic sampling of band-limited signals
  \[
f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t - kh)/2)}{w_s(t - kh)/2}
  \]
- However, it is NOT a causal operator
**Shannon Reconstruction:**

- Let's look at the impulse response:
  \[ h(t) = \frac{\sin(w_s t/2)}{w_s t/2} \]

- The weights are **10%** after 3 samples
  < **5%** after 6 samples

- This construction has a **delay**
  ⇒ **Not good for control**

- Only applied to **periodic sampling**

**Zero-Order Hold (ZOH) & First-Order Hold (FOH)**

- **They are **causal** operators**

- **Predictive FOH:**
  - It is **NOT causal**
    But, can be replaced by **model prediction**

**Sinusoidal signal with h = 1 and h = 0.5**

**Aliasing or Frequency Folding (7.4)**

- **Aliasing:**
  - Two signals with frequency, 0.1 Hz and 0.9 Hz
  - They have the **same values** at all sampling instants
Fourier transform of sampled signal:

\[ F(w) = \int_{-\infty}^{\infty} e^{-jwt} f(t) dt \]

\[ F_s(w) = \sum_{k=-\infty}^{\infty} f(kh)e^{-ikhw} \]

Example 7.1: Feed-water heater in a ship boiler

\[ w_s = \frac{2\pi}{2} = 3.142 \text{ rad/min} \]

\[ w_0 = \frac{2\pi}{2.11} = 2.978 \text{ rad/min} \]

\[ w_s - w_0 = 0.1638 \text{ rad/min} \]

\[ T_s = 38 \text{ min} \]

Frequency Folding

Pre-Sampling Filter in Example 7.2:

Sample + hold of (a) (1 Hz)

Sample + hold of (b)
### Aliasing or Frequency Folding

#### Pre-Sampling Filter in Example 7.2:

- With a sinusoidal perturbation (0.9Hz)
- Sampling frequency = 1 Hz

#### Post-Sampling Filter:

- Because signal from D/A is piecewise constant
  - May excite some oscillatory modes
  - So, use higher-order hold! such as piecewise linear signal

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### Timing Analysis

#### Real-Time Control Systems:
- Computing, Communication, and Control

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Controller</th>
<th>Actuator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_1)</td>
<td>(u_2)</td>
<td>(y_1)</td>
</tr>
<tr>
<td>(u_2)</td>
<td>(u_2)</td>
<td>(y_2)</td>
</tr>
<tr>
<td>(y_1, y_2)</td>
<td>(y_1, y_2)</td>
<td>(y_1, y_2)</td>
</tr>
</tbody>
</table>

Actuator delays:

Sensor delays:

Controller delays:

Plant:

Reference Input:

Sampling instants:

Controller:

Actuator:

02/24/04
Timing Analysis

1 Sensor

1 Controller

1 Actuator

sampling instants

1 Controller

... (k-1)T (k+1)T

k

1 Actuator

R Sensors (-1st, 2nd, ... Rth)

M Actuators (-1st, 2nd, ... Mth)

a1 (k)
a1 (k-1)
a2 (k)
aM (k)
a2 (k-1)

Further Readings

- Multiple time-delay systems
- Random time delays

J. Nilsson, B. Bernhardsson, B. Wittenmark, Lund Institute of Technology

Further Readings

- Multiple time-delay systems
- Random time delays

Multi-Rate Sampling (7.9)

- **Multi-rate System:**

- **Switch Decomposition:**

Further Readings

- **Multi-rate systems**