Introduction

- Time-Domain & Frequency-Domain Characterization:

\[ h(t) \leftrightarrow \mathcal{F} \rightarrow H(j\omega) \]
\[ h[n] \leftrightarrow \mathcal{F} \rightarrow H(e^{j\omega}) \]
## Fourier Series, Fourier Transform, Laplace Transform, z-Transform

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### Fourier Series (FS)

- **Time Domain**: Discrete signals
- **Frequency Domain**: Continuous spectrum

### Fourier Transform (FT)

- **Time Domain**: Continuous signals
- **Frequency Domain**: Continuous spectrum

### Laplace Transform (LT) / z-Transform (zT)

- **Time Domain**: Continuous and discrete signals
- **Frequency Domain**: Continuous and discrete spectrum
Outline

- Representation of a Continuous-Time Signal by its Samples: The Sampling Theorem
- Reconstruction of a Signal from its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals
The Sampling Theorem

- Representation of CT Signals by its Samples

\[ x_1(t) = \mathcal{R}(t) \in \mathbb{R} \]

\[ x_2(t) = \mathcal{R}(t) \in \mathbb{R} \]

\[ x_3(t) = \mathcal{R}(t) \in \mathbb{R} \]
The Sampling Theorem

- Representation of CT Signals by its Samples

\[ x_1(t) \]

\[ x_2(t) \]

\[ x_3(t) \]

\[ x_1(kT) \]

\[ x_2(kT) \]

\[ x_3(kT) \]
The Sampling Theorem

- Representation of CT Signals by its Samples
The Sampling Theorem

\textbf{Representation of CT Signals by its Samples}

\[ x_1(t) \neq x_2(t) \neq x_3(t) \]

\[ x_1(kT) = x_2(kT) = x_3(kT) \]
The Sampling Theorem

- **Impulse-Train Sampling:**

  \[ p(t) : \text{sampling function} \]

  \[ T : \text{sampling period} \]

  \[ w_s = \frac{2\pi}{T} : \text{sampling frequency} \]

  \[ x_p(t) = x(t)p(t) \]

  \[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

  \[ x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \]
The Sampling Theorem

- **Impulse-Train Sampling:**

\[ x_p(t) = x(t) p(t) \quad \xrightarrow{\mathcal{F}} \quad X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(\omega - kw_S)) \]

From multiplication property,

\[ x(t) \quad \xrightarrow{\mathcal{F}} \quad X(j\omega) \]

\[ p(t) \quad \xrightarrow{\mathcal{F}} \quad P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - kw_S) \]

Ex 4.8, pp. 299-300

Ex 4.21, p. 323
Impulse-Train Sampling:

- $w_s > 2w_M$
- $w_s < 2w_M$

Ex 4.21, 4.22, pp. 323-4
The Sampling Theorem:

\[ x(t) : \text{a band-limited signal} \]

with \( X(j\omega) = 0 \) for \(|\omega| > w_M\)

if \( w_s > 2w_M \) where \( w_s = \frac{2\pi}{T} \)

\[ \Rightarrow x(t) \text{ is uniquely determined by } x(nT), n = 0, \pm 1, \pm 2, \ldots, \]

\[ \Rightarrow 2w_M : \text{Nyquist rate} \]

\[ w_M : \text{Nyquist frequency} \]
The Sampling Theorem

- **Exact Recovery by an Ideal Lowpass Filter:**

\[
p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)
\]

![Diagram showing the sampling process and frequency response](image)

- The input signal \( x(t) \) is sampled at a rate \( T \), creating the periodic signal \( p(t) \).
- The frequency response of the ideal lowpass filter is shown, with the ideal lowpass filter \( H(j\omega) \).
- The filtered signal \( x_p(t) \) is the output of the ideal lowpass filter.
- The recovery process is indicated by the signal \( x_r(t) \).

![Diagram showing frequency components](image)

- The frequency components are shown with 
  - \( \omega_s \): Sampling frequency
  - \( \omega_M \): Maximum frequency
  - \( \omega_c \): Critical frequency
- The frequency axis is labeled from \(-\infty \) to \(+\infty\).
The Sampling Theorem

- **Exact Recovery by an Ideal Lowpass Filter:**
The Sampling Theorem

- **Sampling with Zero-Order Hold:**
The Sampling Theorem

- **Sampling with Zero-Order Hold:**

  \[ x(t) = \begin{cases} 
  1, & |t| < T_1 \\
  0, & |t| > T_1 
\end{cases} \]

  \[ X(jw) = \frac{2\sin(wT_1)}{jw} \]

  \[ x(t-t_0) \xrightarrow{\mathcal{F}} e^{-jwt_0} X(jw) \]

  \[ H_0(jw) = e^{-jwT/2} \left[ \frac{2\sin(wT/2)}{w} \right] \]
The Sampling Theorem

- **With Ideal Lowpass Filter & with Zero-Order Hold:**
The Sampling Theorem

- **Sampling with Zero-Order Hold:**

\[ H_0(jw) = e^{-jwT/2} \left[ \frac{2 \sin(wT/2)}{w} \right] \]

\[ H(jw) = H_0(jw)H_r(jw) \]

\[ \Rightarrow H_r(jw) = \frac{e^{jwT/2}H(jw)}{2 \sin(wT/2)} \]
Outline

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Reconstruction of a Signal from its Samples Using Interpolation

- **Exact Interpolation:**

\[
p(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t - nT)
\]

\[
x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT) \delta(t - nT)
\]

\[
x_r(t) = x_p(t) * h(t)
\]

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) h(t - nT)
\]

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin(w_c(t - nT))}{w_c(t - nT)} = X(f)
\]

Ideal lowpass filter with a magnitude of \( T \)

\[
h(t) = T \frac{w_c}{\pi} \frac{\sin(w_c t)}{w_c t}
\]

Ex 2.11, p. 110
Reconstruction of a Signal from its Samples Using Interpolation

- **Exact Interpolation:**

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin(w_c(t - nT))}{w_c(t - nT)}
\]

\[
\frac{w_c}{\pi w_s} \sin \left( \frac{\pi (w_c(t - nT))/\pi}{\pi w_c(t - nT)/\pi} \right)
\]

\[
\frac{2w_c}{w_s} \sin \left( \frac{w_c(t - nT)/\pi}{\pi} \right)
\]
Reconstruction of a Signal from its Samples Using Interpolation
Reconstruction of a Signal from its Samples Using Interpolation

- **Ideal Interpolating Filter & The Zero-Order Hold:***

\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

\[ x(t) \rightarrow \times \rightarrow X_p(j\omega) \rightarrow H(j\omega) \rightarrow x_s(t) \]

\[ h_0(t) \]

\[ x(t) \rightarrow \times \rightarrow X_p(t) \rightarrow x_0(t) \]

\[ H_0(j\omega) = e^{-j\omega T/2} \left[ \frac{2 \sin(\omega T/2)}{\omega} \right] \]
Reconstruction of a Signal from its Samples Using Interpolation

- **Sampling & Interpolation of Images:**

  original image  impulse sampling  zero-order hold  zero-order hold  4 : 1
Reconstruction of a Signal from its Samples Using Interpolation

- **Higher-Order Holds:**

\[ H_1(jw) = \frac{1}{T} \left[ \frac{\sin(wT/2)}{w/2} \right]^2 \]
**Higher-Order Holds:**

\[
x(t) = \begin{cases} 
1, & |t| < T_1 \\
0, & |t| > T_1 
\end{cases}
\]

\[
X(jw) = 2 \frac{\sin(wT_1)}{w}
\]

\[
h_{sq}(t) = \begin{cases} 
1, & -T/2 < t < T/2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
h_1(t) = \frac{1}{T} h_{sq}(t)
\]

\[
H_1(jw) = \frac{1}{T} \left[ \frac{\sin(wT/2)}{w/2} \right]^2
\]
First-Order Hold on Image Processing:

- **zero-order hold**
- **first-order hold**
Reconstruction of a Signal from its Samples Using Interpolation

- **Three Filters: Ideal Lowpass, Zero-Order, First-Order**

**Ideal Lowpass**

- Frequency requirements: \( \omega_M < \omega_c < (\omega_s - \omega_M) \)

**Zero-Order Hold**

- Transfer function: \( H_0(j\omega) = e^{-j\omega T/2} \left[ \frac{2\sin(\omega T/2)}{\omega} \right] \)

**First-Order Hold**

- Transfer function: \( H_1(j\omega) = \tau \left[ \frac{\sin(\omega T/2)}{\omega/2} \right]^2 \)
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Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing

\[ w_s > 2w_M \]  
\[ w_s < 2w_M \]
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing

\[ x(t) = \cos(w_0 t) \]
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing

\[ x(t) = \cos(w_0t) \]
Effect of Under-sampling: Aliasing

- **Overlapping in Frequency-Domain: Aliasing**

\[ x(t) = \cos(w_0 t) \]

\[ w_s > 2w_0 \]

\[ -w_s/2 \quad -w_0 \quad w_0 \quad w_s/2 \]

\[ w_s > 2w_0 \]

\[ -w_s \quad -w_0 \quad w_0 \quad w_s - w_0 \quad w_s + w_0 \]

\[ w_s < 2w_0 \]

\[ -w_s \quad -w_0 \quad -w_0 \quad -w_s \quad w_s \quad w_s \]

**Aliasing**
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing

\[ w_0 = \frac{w_s}{6} \]
Effect of Under-sampling: Aliasing

**Overlapping in Frequency-Domain: Aliasing**

\[ w_0 = \frac{\omega_s}{6} \]

\[ w_0 = \frac{2\omega_s}{6} \]
Effect of Under-sampling: Aliasing

Overlapping in Frequency-Domain: Aliasing

\[ w_0 = \frac{4w_s}{6} \]

\[ w_s - w_0 = w_s - \frac{4w_s}{6} = \frac{2}{3}w_s \]

\[ w_0 = \frac{5w_s}{6} \]

\[ w_s - w_0 = w_s - \frac{5w_s}{6} = \frac{1}{6}w_s \]
Effect of Under-sampling: Aliasing

- **Strobe Effect:**

\[ w_0 = 100 \text{ rad/sec} \]

\[ w = \pm w_s \pm w_0 \]

\[ = \pm 20, -20 \]

\[ w_s = 120 \text{ rad/sec} \]
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Discrete-Time Processing of CT Signals:

- Conversion to discrete time
- Discrete-time system
- Conversion to continuous time
C/D or A-to-D (ADC) and D/C or D-to-A (DAC):

- **C/D**: continuous-to-discrete-time conversion
- **A-to-D**: analog-to-digital converter
- **D/C**: discrete-to-continuous-time conversion
- **D-to-A**: digital-to-analog converter
Discrete-Time Processing of Continuous-Time Signals

- **C/D Conversion:**

  \[ x_c(t) \xrightarrow{\text{C/D conversion}} x_p(t) \times p(t) \xrightarrow{\text{impulse train}} x_d[n] \]

  \[ X_c(j\omega) \xrightarrow{\text{Conversion of impulse train}} X_p(j\omega) \xrightarrow{\text{discrete-time sequence}} X_d(e^{j\Omega}) \]
**C/D Conversion:**

\[ x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT) \delta(t - nT) \]

**Discrete-Time Processing of Continuous-Time Signals**

Table 4.2, p. 329

\[ \delta(t - t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} \]

\[ X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT} \]

\[ X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n] e^{-j\Omega n} \]

\[ = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\Omega n} \]

\[ X_d(e^{j\Omega}) = X_p \left( j \frac{\Omega}{T} \right) \]

**Conversion of impulse train to discrete-time sequence**

\[ = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) \]

Eq 7.3, 7.6, p. 517

\[ = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c \left( j \left( \frac{\Omega}{T} - k \frac{2\pi}{T} \right) \right) \]
Discrete-Time Processing of Continuous-Time Signals

- **C/D Conversion:**

  - **C/D Conversion Diagram:**
    - Continuous-time signal $x_c(t)$
    - Impulse train $p(t)$
    - Conversion to discrete-time sequence $x_d[n]$

  - **C/D Conversion Example:**
    - $x_c(t)$ for $T = T_1$
    - $x_c(t)$ for $T = 2T_1$

  - Discrete-time signal $x_d[n]$ for $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$
C/D Conversion:

\[
X_c(j\omega) \quad X_p(j\omega) \quad X_d(e^{j\Omega})
\]

\[
X_c(j\omega) \quad X_p(j\omega) \quad X_d(e^{j\Omega})
\]
D/C Conversion:

\[ y_d[n] \rightarrow y_p(t) \rightarrow y_c(t) \]

Conversion of discrete-time sequence to impulse train:

\[ x_d(t) \rightarrow x_d[n] \rightarrow \text{Discrete-time system} \rightarrow y_c(t) \]
Discrete-Time Processing of Continuous-Time Signals

- **Overall System:**

\[ x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT) \delta(t - nT) \]

\[ X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT} \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - kw_s)) \]

\[ X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n] e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\Omega n} \]

\[ \Rightarrow X_d(e^{j\Omega}) = X_p \left( j \frac{\Omega}{T} \right) \]
Discrete-Time Processing of Continuous-Time Signals

- Frequency-Domain Illustration:

\[ X_c(j\omega) \]

\[ X_p(j\omega) \]

\[ X_d(e^{j\Omega}) \]

\[ Y_d(e^{j\Omega}) = 1 \cdot X_d(e^{j\Omega}) \]

\[ Y_p(j\omega) = H_p(j\omega)X_p(j\omega) \]

\[ Y_c(j\omega) = H_c(j\omega)X_c(j\omega) \]
Discrete-Time Processing of Continuous-Time Signals

CT & DT Frequency Responses:

\[ Y_C(j\omega) = X_C(j\omega)H_C(j\omega) \]

\[ H_C(j\omega) = \begin{cases} 
H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\
0, & |\omega| > \omega_s/2 
\end{cases} \]
Discrete-Time Processing of Continuous-Time Signals

- Frequency-Domain Illustration:

\[ X_c(j\omega) \]

\[ Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega}) \]

\[ X_p(j\omega) \]

\[ Y_p(j\omega) = H_p(j\omega)X_p(j\omega) \]

\[ X_d(e^{j\Omega}) \]

\[ Y_c(j\omega) = H_c(j\omega)X_c(j\omega) \]
**Digital Differentiator:** (band-limited)

\[
H_c(j\omega) = \begin{cases} 
    j\omega, & |\omega| < \omega_c \\
    0, & |\omega| > \omega_c 
\end{cases}
\]

\[
H_d(e^{j\Omega}) = j \left( \frac{\Omega}{T} \right), \quad |\Omega| < \pi
\]

\[
\Omega = \frac{\omega}{T}, \quad \omega_s = 2\omega_c
\]
### Delay: (band-limited)

Ex 4.15, p. 317

\[
H_c(jw) = \begin{cases} 
  e^{-jw\Delta}, & |w| < w_c \\
  0, & |w| > w_c 
\end{cases}
\]

\[H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi\]

\[\Omega = wT, \quad w_s = 2w_c\]
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- Sampling of Discrete-Time Signals
Impulse-Train Sampling:

\[ x_p[n] = \begin{cases} 
  x[n], & \text{if } n = kN \\
  0, & \text{otherwise}
\end{cases} \]

\[ x_p[n] = x[n] p[n] \]

\[ = \sum_{k=-\infty}^{+\infty} x[kN] \delta[n - kN] \]
Sampling of Discrete-Time Signals

- **Impulse-Train Sampling:**

\[
P(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\Omega - kw_s)
\]

\[
X_p(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} P(e^{j\theta})X(e^{j(\Omega-\theta)})d\theta
\]

\[
\Rightarrow X_p(e^{j\Omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\Omega-kw_s)})
\]
Exact Recovery Using Ideal Lowpass Filter:

\[ X(e^{j\Omega}) \]

\[ X_p(e^{j\Omega}) \]

\[ H(e^{j\Omega}) \]

\[ X_r(e^{j\Omega}) \]
DT Decimation & Interpolation: Down-sampling

Eq 5.45, p. 378: Time expansion

\[ X_b(e^{j\Omega}) = X_p(e^{j\Omega/N}) \]
Higher Equivalent Sampling Rate: Up-sampling
**Down-sampling + Up-sampling:**

\[
\frac{2\pi}{9} \times 4 < \pi
\]

\[
\frac{2\pi}{9} \times \frac{9}{2} = \pi
\]

\[
\frac{2\pi}{9} \times \frac{1}{2} = \frac{\pi}{9}
\]

\[
\frac{\pi}{9} \times 9 = \pi
\]
The Sampling Theorem

- Exact Recovery by an Ideal Lowpass Filter:
Reconstruction of a Signal from its Samples Using Interpolation

**Exact Interpolation:**

\[
x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \cdot \frac{w_c T}{\pi} \cdot \frac{\sin(w_c(t - nT))}{w_c(t - nT)}
\]

\[
= \frac{w_c}{\pi w_s} \cdot 2\pi \cdot \frac{\sin\pi(w_c(t - nT)/\pi)}{\pi w_c(t - nT)/\pi}
\]

\[
= \frac{2w_c}{w_s} \cdot \text{sinc}(\theta)
\]

\[
\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}
\]
Three Filters: Ideal Lowpass, Zero-Order, First-Order

- Ideal lowpass

- Zero-order hold

- First-order hold
Effect of Under-sampling: Aliasing

- Overlapping in Frequency-Domain: Aliasing

\[ x(t) = \cos(w_0t) \]
In Summary

- **Discrete-Time Processing of CT Signals**

![Diagram of discrete-time processing of CT signals]
CT & DT Frequency Responses:

\[ Y_c(j\omega) = X_c(j\omega)H_c(j\omega) \]

\[ H_c(j\omega) = \begin{cases} 
    H_d(e^{j\omega T}), & |\omega| < w_s/2 \\
    0, & |\omega| > w_s/2 
\end{cases} \]
Decimation (Down-sampling): 

Interpolation (Up-sampling):
Problem 7.20 (p.560):

(a) SA: Inserting one zero after each sample
(b) SB: Decimation 2:1, extracting every second sample

Which corresponds to low-pass filtering with $\omega_c = \pi/4$?
Problem 7.20 (p.560):

(a)
**Problem 7.20 (p.560):**

(b)
Problem 7.23 (p.562):

\[ p(t) = p_1(t) - p_1(t - \Delta) \]
Problem 7.23 (p.562):
In Summary

\[ h(t) \xrightarrow{\text{C.I.F.T.}} H(j\omega) \]

\[ h(t) \xrightarrow{\text{D.T.F.T.}} h[n] \xrightarrow{\text{D.T.F.T.}} H(e^{j\Omega}) \]

\[ \omega_s = \frac{2\pi}{T} \]

\[ \Omega = wT \]
In Summary

- **Discrete-Time Processing of CT Signals**

Diagram illustrating the processing of continuous-time signals through discrete-time operations, including sampling, impulse train conversion, and frequency response.
Chapter 7: Sampling

- Representation of a CT Signal by Its Samples:
  - The Sampling Theorem

- Reconstruction of a Signal from Its Samples
  - Using exact interpolation
  - Using zero-order hold
  - Using higher-order hold

- The Effect of Under-sampling
  - Overlapping in Frequency-Domain
  - Aliasing

- DT Processing of CT Signals

- Sampling of Discrete-Time Signals
  - Down-sampling
  - Up-sampling
Flowchart

Introduction (Chap 1)

Bounded/Convergent

Periodic

FS

CT

DT

(Chap 3)

Aperiodic

CT

FT

DT

(Chap 4)

(Chap 5)

Unbounded/Non-convergent

LT

CT

(Chap 9)

zT

DT

(Chap 10)

Time-Frequency (Chap 6)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing ( DSP-8 )