Capacitance of Microstrip Lines with Inhomogeneous Substrate

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Abstract—A mode-matching approach combined with Galerkin's method is proposed in this paper to calculate the capacitance matrix of microstrip lines embedded in an inhomogeneous stratified medium. Eigenmodes in each layer is first solved numerically, and the potential in each layer can be expressed in terms of these eigenmodes. Coupling between two sets of eigenmodes in contiguous layers are described by defining reflection matrices. A Green’s function is thus obtained in terms of these eigenmode sets to relate the potential to a line charge. Integral equation is then constructed relating the charge distribution and the imposed voltage on the microstrip surface. Galerkin’s method is next applied to solve the charge distribution and hence the capacitance matrix. Several inhomogeneous profiles are studied to understand the effects of inhomogeneities on the capacitance and relevant parameters.

I. INTRODUCTION

For a microstrip deposited at the interface between a dielectric and free space, approximate closed form for capacitance is plausible [1]. Conformal mapping technique has been applied to calculate the capacitance matrix of several microstrips lying in the same plane enclosed by a rectangular conducting box [2].

For several microstrip lines embedded in different layers of a stratified medium, numerical methods are often resorted. In [3]–[5], a spatial domain approach using the free space Green’s function has been developed to calculate the capacitance and the inductance matrices of multiconductor transmission lines located arbitrarily in a multilayered medium of finite extent. The potential in the medium is expressed in terms of the charge distribution and hence the capacitance matrix. Several inhomogeneous profiles are studied to understand the effects of inhomogeneities on the capacitance and relevant parameters.

II. FORMULATION

In Fig. 1, we show the configuration of a line charge source in layer (m) of a stratified medium. The whole structure is uniform in the y direction. In each layer, the dielectric constant is a piecewise continuous function of x and is independent of z. Two perfect electric conductor walls are located at x = 0 and x = a as the potential reference.

In the electrostatic (EQS) limit [13], potential in layer (m) in the absence of charge source is obtained by solving the following Laplace’s equation:

$$\left(\epsilon_m(x) \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \phi(x, z) = 0. \quad (1)$$

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By separation of variables, \( \phi(x, z) \) can be expressed as a product of \( \psi(x) \eta(z) \), and (1) is reduced to

\[
\begin{align*}
\epsilon_m^{-1}(x) \frac{d}{dx} \epsilon_m(x) \frac{d}{dx} \psi(x) &= -k^2 \psi(x) \\
\frac{d^2}{dz^2} \eta(z) &= k^2 \eta(z)
\end{align*}
\]

(2)

Next, choose a set of basis functions \( S_p(x) = \sqrt{2/a} \sin(\alpha_p x) \) with \( \alpha_p = \pi n / a \). These basis functions have orthonormal properties that \( \langle S_p(x), S_q(x) \rangle = \delta_{p q} \) where the inner product is defined over the interval \([0, a]\). Expand the \( n \)th eigensolution \( \psi_n(x) \) by these basis functions as \( \psi_n(x) = \sum_{p=1}^{N} c_{p n} S_p(x) \), and substitute it into (2). Take the inner product of \( S_p(x) \) with the resulting equation and apply the orthonormality property of \( S_p(x)'s \) to obtain

\[
\sum_{p=1}^{N} \langle S'_p(x), \epsilon_m(x) S_p(x) \rangle c_{p n} = k^2_n \sum_{p=1}^{N} \langle S_p(x), \epsilon_m(x) S_p(x) \rangle b_{n p}, \quad 1 \leq q \leq N.
\]

(3)

From (3), \( N \) eigenvalues \( k^2_n \) and their associated eigenvectors \( b_n \) can be obtained. These eigensolutions are normalized to have

\[
\langle \psi_m(x), \epsilon_m(x) \psi_n(x) \rangle = b_m^* \Gamma_m b_n
\]

(4)

where the \((q, p)\)th element of \( \Gamma_m \) is \( \langle S_q(x), \epsilon_m(x) S_p(x) \rangle \).

Next, consider a line charge with density \( \rho_0 \) located at \((z_0, x_0)\) in layer \((m)\) of the stratified medium. In the absence of other layers, the potential is obtained by solving the Poisson’s equation

\[
\left( \epsilon_m^{-1}(x) \frac{\partial}{\partial x} \epsilon_m(x) \frac{\partial}{\partial x} + \frac{\partial^2}{\partial z^2} \right) \phi(x, z) = -\frac{\rho_0}{\epsilon_m(x_0)} \delta(x - x_0) \delta(z - z_0).
\]

(5)

The solution \( \phi(x, z) \) can be expressed in terms of the eigensolutions \( \psi_p(x) \) as

\[
\phi(x, z) = \sum_{p=1}^{N} \frac{\rho_0}{2\epsilon_m(x_0)} \psi_p(x) \psi_p(x_0) e^{-k_p |x - x_0|}
\]

(6)

where \( K_m = \text{diag} \cdot [k_1, k_2, \ldots, k_N], \quad e^{-K_m |x - x_0|} = \text{diag} \cdot [e^{-k_1 |x - x_0|}, e^{-k_2 |x - x_0|}, \ldots, e^{-k_N |x - x_0|}] \), and
\( \psi'_m(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_N(x)] \) are eigensolutions in layer \((m)\).

In the presence of other inhomogeneous layers as in Fig. 1, the potential in layer \((m)\) can be expressed as

\[
\phi_m(x, z) = \psi'_m(x) \left[ e^{-K_{m+1} z_m} A_m + e^{-K_m z_m} B_m + \frac{\rho_0}{2} K_m e^{-K_m |z_m - z'_m|} \psi_m(x_0) \right]
\]

(7)

where \( z_m = z + d_m \) and \( z'_m = z_0 + d_m \). The first term in the bracket decays in the positive \( z \) direction, and the second term decays in the negative \( z \) direction. At \( z = -d_m \), define a reflection matrix \( R_{cm} \) which relates the upward-decaying potential to the downward-decaying potential as

\[
A_m = R_{cm} \left[ B_m + \frac{\rho_0}{2} K_m e^{-K_m |z_m - z'_m|} \psi_m(x_0) \right].
\]

(8)

Similarly, define another reflection matrix \( R_{cm} \) at \( z = -d_{m-1} \) as

\[
e^{-K_{m-1} z_m} B_m = R_{cm} \left[ e^{-K_{m-1} z_m} A_m + \frac{\rho_0}{2} K_m e^{-K_{m-1} z'_m} \psi_m(x_0) \right].
\]

(9)

From (8) and (9), \( A_m \) and \( B_m \) are solved explicitly as

\[
A_m = \frac{\rho_0}{2} \left[ I - R_{cm} \cdot e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} \right]^{-1} R_{cm} \cdot e^{-K_{m-1} z_m} A_m + \frac{\rho_0}{2} K_m e^{-K_{m-1} z'_m} \psi_m(x_0)
\]

\[
B_m = \frac{\rho_0}{2} \left[ I - e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} \right]^{-1} e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} B_m + \frac{\rho_0}{2} K_m e^{-K_{m-1} z'_m} \psi_m(x_0).
\]

(10)

The potential in layer \((m)\) can thus be written in a compact form as

\[
\phi_m(x, z) = \rho_0 \psi'_m(x) \left[ T_{mm}(z_m, z'_m) \cdot \psi_m(x_0) \right]
\]

(11)

where

\[
T_{mm}(z_m, z'_m) = \frac{1}{2} \left\{ e^{-K_{m+1} z_m} \left[ \left[ I - R_{m} \cdot e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} \right] \right]^{-1} \cdot R_{cm} \cdot e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} \cdot e^{-K_{m+1} z'_m}, \right.
\]

\[
\left. \cdot \left[ I - R_{cm} \cdot e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} \right]^{-1} \cdot R_{cm} \cdot e^{-K_{m-1} z_m} R_{cm} \cdot e^{-K_{m-1} z'_m} + K_m e^{-K_{m+1} z'_m} \psi_m(x_0) \right\}.
\]

(12)

The potential in layer \((l)\) with \( l < m \) can be expressed as

\[
\phi_l(x, z) = \psi'_l(x) \cdot \left[ e^{-K_{l+1} z_l} + e^{-K_{l-1} z_{l-1}} R_{cl} \cdot e^{-K_{l-1} z_{l-1}} \right] A_l.
\]

(13)

Imposing the boundary condition that potential is continuous at \( z = -d_{m-1} \), we have

\[
\psi'_m(x) \cdot \left[ I + e^{-K_{m-1} z_m} e^{-K_{m-1} z'_m} \right] R_{cm} \cdot e^{-K_{m-1} z'_m} \cdot \psi_m(x_0) = \rho_0 \psi'_m(x) \cdot T_{mm}(h_m, z'_m) \cdot \psi_m(x_0).
\]

(14)
Take the inner product of \( e_{m-1}(z) \psi_{m-1}(x) \) with (14) to obtain

\[
A_{m-1} = \rho_0 \left[ \frac{I + e^{-K_{m-1}h_{m-1}}}{\bar{H}_{(m-1)m}} \cdot \frac{\bar{R}_{(m-1)}}{T_{mm}} \right] (h_m, z_m) \cdot \bar{\psi}_m(x_0) \]

where \( \bar{H}_{qp} = \langle \bar{\psi}_q(x), \psi_p(x) \rangle \). Imposing the boundary condition that the potential and the normal electric flux density are continuous at \( z = -d_i \), we have

\[
\psi_1^t(x) \cdot \left[ \frac{I + e^{-K_{h_1}}}{\bar{R}_{ij}} \cdot e^{-K_{h}} \right] \cdot \bar{\psi}_i = \psi_{i+1}^t(x) \cdot \left[ \frac{I + \bar{R}_{ij}(l+1)}{e^{-K_{h}} \bar{R}_{ij}} \right] \cdot \bar{\psi}_i
\]

(16)

\[
\epsilon_i(x) \psi_i^t(x) = K_i \cdot \left[ I - \frac{1}{\bar{R}_{ij}(l+1)} \cdot e^{-K_{h}} \bar{R}_{ij} \right] \cdot \bar{\psi}_i
\]

(17)

Take the inner product of \( \epsilon_i(x) \bar{\psi}_i(x) \) with (16) to have

\[
\bar{A}_i = \left[ I + e^{-K_{h}} \bar{R}_{ij} \cdot e^{-K_{h}} \bar{R}_{ij} \right] \cdot \bar{H}_{ij} \cdot \bar{R}_{ij}(l+1)
\]

(18)

Take the inner product of \( \psi_i(x) \) with (17) to have

\[
K_i \cdot \left[ I - \frac{1}{\bar{R}_{ij}(l+1)} \cdot e^{-K_{h}} \bar{R}_{ij} \right] \cdot \bar{H}_{ij} \cdot \bar{R}_{ij}(l+1)
\]

(19)

From (18) and (19), we obtain the recursive relation between the reflection matrices as

\[
\frac{\bar{R}_{ij}(l+1)}{\bar{R}_{ij}(l+1)} = \left\{ \left[ I - e^{-K_{h}} \bar{R}_{ij} \cdot e^{-K_{h}} \bar{R}_{ij} \right] \cdot K_i \cdot \frac{1}{\bar{R}_{ij}(l+1)} \cdot \bar{H}_{ij} \cdot \bar{R}_{ij}(l+1) \right\}^{-1}
\]

(20)

Similarly, the potential in layer \( (l) \) with \( l > m \) can be expressed as

\[
\phi_i(x, z) = \psi_i^t(x) \cdot \left[ e^{-K_{i-1}h_{i-1}} \bar{R}_{ij} + e^{K_{i-1}h_{i-1}} \right] \cdot \bar{B}_i
\]

(21)

Imposing the boundary condition that potential is continuous at \( z = -d_i \), we have

\[
\bar{B}_{m+1} = \rho_0 \left[ e^{-K_{m+1}h_{m+1}} \bar{R}_{ij}(m+1) + e^{K_{m+1}h_{m+1}} \right] \cdot \bar{H}_{(m+1)m} \cdot T_{mm} \cdot (0, z_m) \cdot \psi_m(x_0)
\]

(22)


dimensions: 614.0 x 793.9

III. NUMERICAL RESULTS

In Fig. 2, the effective dielectric constant of a microstrip line is shown as a function of the relative permittivity of the substrate under the inhomogeneous layer. The results agree reasonably with those in [12].

Next, two periodical substrate structures similar to that in [10] is modeled. In Fig. 3, the capacitance of the microstrip
line is shown as a function of lateral shift of the periodical substrate relative to the strip. In the substrate, the shaded areas represent glass fiber and the white areas represent epoxy. As more glass fiber is laterally shifted underneath the strip, the capacitance increases. The variation range is smaller for the structure with three interleaving layers. Because the mixture of epoxy in between fiber glass areas reduces the relative permittivity underneath the strip, the shift of substrate produces less significant permittivity change than the other configuration.

In Fig. 4, we present the effective dielectric constant of a microstrip line with an air pocket underneath the supporting film. The air pocket is observed to increase the phase velocity of the signal along the strip, and the effect is more significant with a larger pocket.

On the other hand, the phase velocity may be reduced by enclosing the strip with a high permittivity material. As shown in Fig. 5, the larger the area with high permittivity material, the slower the quasi-TEM signal will propagate. In both Figs. 4 and 5, the effective dielectric constant converges to that with a homogeneous substrate as $d$ approaches zero.

Heat treatment is applied in certain applications when depositing microstrips onto the substrate, which may cause permittivity variation in the substrate right underneath the strip. In Fig. 6, we show the capacitance of two coupled microstrip lines with permittivity variation in the area underneath the strips. Both the self and mutual capacitance increase with
increasing permittivity. The capacitances with $\varepsilon_r = 12$ are those with a homogeneous substrate.

In Fig. 7, the effective dielectric constant of two coupled microstrip lines attached to a corrugated substrate surface is presented. The corrugation is used to model surface roughness. The effective dielectric constants of the even and the odd modes are defined as

$$\varepsilon_{\text{eff}}^{(e)} = \frac{C_{11} + C_{12}}{C_{11}^{(0)} + C_{12}^{(0)}}, \quad \varepsilon_{\text{eff}}^{(o)} = \frac{C_{11} - C_{12}}{C_{11}^{(0)} - C_{12}^{(0)}}$$

where $C_{11}$ and $C_{12}$ are the self and mutual capacitance per unit length of two coupled strips embedded in inhomogeneous substrates, and $C_{11}^{(0)}$ and $C_{12}^{(0)}$ are the self and mutual capacitance per unit length of the strips with all dielectrics replaced by free space.

The effective dielectric constant of the even mode is higher than that of the odd mode. Either increasing the corrugation depth $d$ or period $P$ reduces $\varepsilon_{\text{eff}}$ because more free space is mixed with the substrate near the strips.

Next, consider protrusion or indentation of the substrate. In Fig. 8, both the self and mutual capacitances increase when the substrate is indented and the strip is plated at the bottom of the dent. In such case, the electric field tends to concentrate near the bottom of the configuration, which has a higher permittivity. When the strip is deposited on top of a protrusion, part of the electric field remains in the free space, and the capacitance decreases. The capacitances with $d = 0$ are those with a homogeneous substrate.

In Fig. 9, we show the effective dielectric constant of the odd mode for two coupled microstrip lines deposited on a film with an air pocket beneath. The phase velocity increases when the depth or width of the pocket is enlarged. Similar observations can be found with the even mode as shown in Fig. 10. In both figures, the effective dielectric constants tend to converge to those with a homogeneous substrate, but the convergence is not as obvious as that for the single strip case shown in Figs. 4 and 5.

Finally in Figs. 11 and 12, we show the effective dielectric constants of two coupled microstrip lines with a high permittivity material under them. The effective dielectric constant increases as the depth or width of the high permittivity area is enlarged. The convergence to the effective dielectric constant with a homogeneous substrate is more obvious than that in Figs. 9 and 10.

All the results demonstrate that this technique can be applied to model microstrip lines with more complicated substrates such as with periodical loading of glass fibers (to enhance...
the substrate's mechanical strength), air pocket (to increase the phase velocity), high permittivity enclosure (to decrease the phase velocity), permittivity variation due to thermal treatment, surface roughness, protrusion or dent (to reduce or enhance coupling). Other structures occurring in applications with similar configuration can also be solved by using this technique.

IV. CONCLUSION

A mode-matching approach combined with Galerkin's method is proposed in this paper to calculate the capacitance matrix of microstrip lines embedded in an inhomogeneous stratified medium. The kernel of the integral equation is expressed in terms of eigenmodes solved numerically. Galerkin's method is applied to solve the charge distribution and hence the capacitance matrix. Several inhomogeneous dielectric profiles are studied to understand the effects of inhomogeneities on the capacitance and other related parameters like effective dielectric constant and phase velocity.

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REFERENCES


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