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REFERENCES


Axisymmetric Modes of Cylindrical Resonators with Cascaded Inhomogeneous Dielectrics

Jean-Fu Kiang

Abstract—A generic numerical scheme is developed to calculate the resonant frequency of axisymmetric modes in an inhomogeneous cylindrical dielectric resonator. The resonator consists of sections of cylindrically stratified dielectrics within a cylindrical waveguide. In each section, the TMom and TEMom waveguide modes are solved by expanding the \( H_\phi \) and \( E_z \) components in terms of the eigenmodes in an empty waveguide. The fields in each section are then expanded in terms of these TMom and TEMom modes. The transverse resonance technique is then applied to obtain the resonant frequencies. Comparison with literatures validates the effectiveness of this approach. Results with continuous dielectric profiles are also obtained.

I. INTRODUCTION

Cylindrical cavities have been used to cure materials [1], to measure complex permittivity of materials [2], and as a resonator in microwave circuits [3]-[11]. In all these applications, the circular waveguide section forming the cavity contains inhomogeneous dielectrics. For the resonator application, the resonant frequencies of the loaded cavity need to be determined precisely. Several coupled dielectric rod or ring resonators can be arranged coaxially in a circular waveguide to form a bandpass filter. The resonant frequencies of the axisymmetric TEM01 and TEM10 modes have been calculated by using a mode-matching technique [3], [4]. For both modes, the resonant frequencies are below the cutoff frequency of the TEM01 waveguide mode. In [5] and [6], the resonant frequency of nonaxisymmetric hybrid modes are calculated by using a similar technique.

In [7], a finite integration technique (FIT) based on the integral forms of Maxwell's equations is proposed to calculate the resonant frequencies of a cavity filled with an inhomogeneous dielectric. A brief summary of mode nomenclature is also provided in [7]. In [8], a variational expression is used to calculate the resonant frequencies of axisymmetric modes where the radial variation of field components are expanded by the first-order finite element (FE) basis functions and the axial variation is expanded in terms of sinusoidal. Finite-difference method in the frequency domain [9], finite-difference time-domain (FDTD) method [10], and finite element method (FEM) [11] have also been used.

Mode-matching method proves to be efficient for many canonical resonator structures. For example, the cylindrical dielectric rod and ring in a cylindrical cavity. Usually, the eigenmodes in a stratified medium need to be solved first to represent the field distribution in the later stage. If the dielectric ring consists of many layers or if a dielectric rod has a continuous permittivity profile, conventional mode-matching method becomes tedious or impossible. For such structures, finite element method, FDTD method, and FIT method can be used at the expense of finer grids to express the fields accurately.

In this paper, we will present a generic numeric scheme to solve such problems. First, the eigenmodes in each uniform dielectric loaded waveguide section are obtained by solving a symmetric eigenvalue problem, where dielectrics with continuous profile can also be handled. Reflection matrices at the junctions of waveguide sections are defined to reduce the number of unknowns. Then the transverse resonance technique is applied to obtain the resonant frequencies of the resonators.

II. FORMULATION

Fig. 1 shows the configuration of a cylindrical resonator with radius \( a \), which consists of several sections of circular waveguides loaded with inhomogeneous dielectrics. The permittivity in each layer is a piecewise continuous functions of \( \phi \) and \( z \).
Axisymmetric modes exist in such a medium, and are categorized into TM \((m = 0)\) and TE \((m \neq 0)\) modes [7]. For the TM modes, the existing field components are \(E_x, E_y,\) and \(H_z\). For the TE modes, the existing field components are \(H_x, H_y,\) and \(E_z\).

First, consider the TM modes in an infinitely long circular waveguide with an inhomogeneous dielectric profile which is uniform along the axial direction. Expand Maxwell’s equations to obtain

\[
\begin{align*}
\rho \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial H_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 H_z}{\partial \phi^2} + k^2 H_z &= 0, \\
\frac{\partial E_z}{\partial z} &= j \omega \epsilon_z H_z,
\end{align*}
\]

Next, express \(H_z\) of the \(n\)th eigenmodes by a set of basis functions \(S_m(\rho)\) as

\[
H_z = \sum_{m=1}^{N} b_m S_m(\rho) e^{\pm ik_m z}.
\]

where \(b_m = [b_{11}, \ldots, b_{NN}]^T,\) and \(S'(\rho) = [S_1(\rho), \ldots, S_N(\rho)]\). Choose the \(H_z\) distribution in an empty circular waveguide as the basis functions, i.e., \(S_m(\rho) = J_0(k_m \rho)\) with \(J_0(k_m) = 0\). Substitute (2) into (1), then take the inner product of \(S_m(\rho)\) with the resulting equation and use the integration by parts technique to have

\[
\sum_{m=1}^{N} b_m \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial S_m(\rho)}{\partial \rho} \right) \right] (S_m(\rho), S_m(\rho)) + (S_m(\rho), \omega^2 \mu \rho S_m(\rho)) = 0
\]

where the inner product is defined over \(0 < \rho < a\). Hence, (3) constitutes a symmetric eigenvalue problem to be solved numerically for the propagation constant \(k_m\).

The first term in \(H_{t\phi}\) and \(E_{r\phi}\) is a wave propagating in the +\(z\) direction, and the second term is a wave propagating in the -\(z\) direction. Define a reflection matrix \(R_{I\rightarrow R}\) at the upper boundary \(z = -d_1\) so that the reflection matrix multiplied by the upward wave gives the downward wave, i.e., \(R_{I\rightarrow R} e^{iR_{I\rightarrow R} z} = e^{-iR_{I\rightarrow R} z} \cdot \beta_{r+1}\). Define another reflection matrix \(R_{II\rightarrow R}\) at the lower boundary \(z = -d_1\) so that the reflection matrix multiplied by the downward wave gives the upward wave, i.e., \(R_{II\rightarrow R} e^{-iR_{II\rightarrow R} z} = \alpha_{r-1}\).

Thus, we obtain the resonance condition

\[
\det \left( \mathbf{I} - R_{I\rightarrow R} e^{iR_{I\rightarrow R} z} \cdot R_{II\rightarrow R} e^{-iR_{II\rightarrow R} z} \right) = 0.
\]

The resonant frequencies are obtained by solving (6).

By matching the boundary conditions that \(H_{t\phi} = H_{(r+1)\phi}\) and \(E_{r\phi} = E_{(r+1)\phi}\) at \(z = -d_r\), we have

\[
\begin{align*}
\left( \mathbf{I} - R_{I\rightarrow R} \right) \cdot \beta_{r+1} &= \left( \mathbf{I} + R_{I\rightarrow R} \right) \cdot \beta_r, \\
\left( \mathbf{I} - R_{II\rightarrow R} \right) \cdot \alpha_{r-1} &= \left( \mathbf{I} + R_{II\rightarrow R} \right) \cdot \alpha_r.
\end{align*}
\]

By matching the boundary conditions that \(H_{t\phi} = H_{(r+1)\phi}\) and \(E_{r\phi} = E_{(r+1)\phi}\) at \(z = -d_r\), we have

\[
\begin{align*}
H_{t\phi} &= \sum_{m=1}^{N} \frac{\partial}{\partial \rho} \left( \frac{\partial S_m(\rho)}{\partial \rho} \right) (S_m(\rho), S_m(\rho)) + (S_m(\rho), \omega^2 \mu \rho S_m(\rho)), \\
E_{r\phi} &= \frac{1}{\omega \epsilon_{r\phi}} \sum_{m=1}^{N} \frac{\partial}{\partial \rho} \left( \frac{\partial S_m(\rho)}{\partial \rho} \right) (S_m(\rho), S_m(\rho)).
\end{align*}
\]
where \( \tilde{B}_{ij} \) is defined as \( \langle \rho e^{-1}(\rho) \tilde{\phi}_i(\rho), \tilde{\phi}_j(\rho) \rangle \). A recursive formula is thus obtained as shown in (9) at the bottom of the previous page.

The fields in layer \((m)\) with \( m < l \) can be expressed as shown in (10) at the bottom of the previous page. Expand Maxwell’s equations in terms of \( H_z, H_\rho, \) and \( E_\phi \). Next, expand the eigenmode of \( E_\phi \) by a set of basis functions \( \tilde{S}_m(\rho) \), and choose the \( E_\phi \) distribution in an empty circular waveguide as the basis functions to expand \( E_\phi \). The eigenmode can be obtained by solving the eigenvalue problem formed by taking the inner product of \( \tilde{S}_\varphi(\rho) \) with the equation satisfied by \( E_\phi \).

Next, use the waveguide modes to represent the fields in layer \((l)\) of the resonator. Reflection matrices \( \tilde{R}_{ul} \) and \( \tilde{R}_{rl} \) are defined to relate the upward wave and the downward wave in layer \((l)\). Finally, the resonant frequencies are obtained by solving the resonance condition

\[
\text{det} \left( \mathbf{I} - \tilde{R}_{rl} \cdot e^{i \tilde{B}_{l+1} h_l} \cdot \tilde{R}_{ul} \cdot e^{-i \tilde{B}_{l+1} h_l} \right) = 0.
\]

Recursive formulas for \( \tilde{R}_{ul} \) and \( \tilde{R}_{rl} \) can be derived by matching \( E_\phi \) and \( H_\rho \) at interfaces between contiguous layers.

### III. Numerical Results

First, we show the resonant frequencies of the TE\(_{01}\), TE\(_{02}\), TM\(_{01}\), and TM\(_{02}\) modes of a cylindrical dielectric-loaded resonator as shown in Table I. The results from [7] are also shown for comparison.

#### TABLE I

<table>
<thead>
<tr>
<th>Mode</th>
<th>( N = 10 )</th>
<th>( N = 15 )</th>
<th>( N = 20 )</th>
<th>( N = 25 )</th>
<th>( N = 30 )</th>
<th>( \text{measured} )</th>
<th>( \text{computed} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE(_{01})</td>
<td>7.02</td>
<td>7.02</td>
<td>7.02</td>
<td>7.02</td>
<td>7.037</td>
<td>6.943</td>
<td></td>
</tr>
<tr>
<td>TE(_{02})</td>
<td>11.39</td>
<td>11.39</td>
<td>11.39</td>
<td>11.39</td>
<td>11.39</td>
<td>11.316</td>
<td></td>
</tr>
<tr>
<td>TM(_{01})</td>
<td>9.54</td>
<td>9.45</td>
<td>9.43</td>
<td>9.42</td>
<td>9.41</td>
<td>9.596</td>
<td>9.185</td>
</tr>
<tr>
<td>TM(_{02})</td>
<td>11.44</td>
<td>11.36</td>
<td>11.27</td>
<td>11.25</td>
<td>11.23</td>
<td>11.113</td>
<td>10.943</td>
</tr>
</tbody>
</table>

The convergence rate for the TE\(_{01}\) and TE\(_{02}\) modes are faster than that for the TM\(_{01}\) and TM\(_{02}\) modes in this case.

Table II shows the resonant frequencies of a dielectric resonator on top of a substrate as in a circuit board environment. Our results compare favorably with those in the literatures. Table III shows the resonant frequencies of a cylindrical dielectric resonator. The results are close to those in the literatures.

Next, we calculate the resonant frequencies of two symmetrically coupled dielectric ring resonators in a circular waveguide. The permittivity of the ring is assumed to have a parabolic profile with an extreme value \( \varepsilon_m \) at \( \rho = (a + b)/2 \). The resonant frequencies

\[
\tilde{R}_{ul} = \left\{ \left[ \mathbf{I} - e^{i \tilde{B}_{l+1} h_l} \cdot \tilde{R}_{ul} \cdot e^{-i \tilde{B}_{l+1} h_l} \right]^{-1} \cdot \tilde{B}_{l+1} \right\}^{-1}
\]

\[
\tilde{R}_{rl} = \left\{ \left[ \mathbf{I} + e^{i \tilde{B}_{l+1} h_l} \cdot \tilde{R}_{ul} \cdot e^{-i \tilde{B}_{l+1} h_l} \right]^{-1} \cdot \tilde{B}_{l+1} \right\}^{-1}
\]

\[
\tilde{R}_{ul} = \left\{ \left[ \mathbf{I} - e^{i \tilde{B}_{l+1} h_l} \cdot \tilde{R}_{ul} \cdot e^{-i \tilde{B}_{l+1} h_l} \right]^{-1} \cdot \tilde{B}_{l+1} \right\}^{-1}
\]

\[
\tilde{R}_{rl} = \left\{ \left[ \mathbf{I} + e^{i \tilde{B}_{l+1} h_l} \cdot \tilde{R}_{ul} \cdot e^{-i \tilde{B}_{l+1} h_l} \right]^{-1} \cdot \tilde{B}_{l+1} \right\}^{-1}
\]

(12)
TABLE III

<table>
<thead>
<tr>
<th>Mode</th>
<th>Computed</th>
<th>Present</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE_{016}</td>
<td>3.426</td>
<td>3.426</td>
<td>3.426</td>
</tr>
<tr>
<td>TE_{026}</td>
<td>5.419</td>
<td>5.412</td>
<td>5.412</td>
</tr>
<tr>
<td>TM_{016}</td>
<td>4.572</td>
<td>4.561</td>
<td>4.542</td>
</tr>
</tbody>
</table>

Finally, we calculate the resonant frequencies of two symmetrically coupled dielectric rod resonators in a circular waveguide. The permittivity of the rod is assumed to have a parabolic profile with an extreme value $\varepsilon_m$ at $\rho = 0$. The resonant frequencies are below the cutoff frequency of the circular waveguide. As shown in Fig. 3, the results with a flat profile in the rod match well with those in [4]. The resonant frequency decreases as $\varepsilon_m$ increases. The difference between $f_{oe}$ and $f_{om}$ increases as the two resonators move closer to each other. Note that $f_{oe}$ is lower than $f_{om}$ in this case, and $f_{oe}$ is higher than $f_{om}$ for the coupled dielectric rings in the previous case.

IV. CONCLUSION

A general numeric scheme combining the eigenvalue method and the transverse resonance technique has been developed to calculate the resonant frequencies of a cylindrical resonator consisting of cascaded sections of circular waveguides loaded with inhomogeneous dielectrics. The results obtained by using this approach compare favorably with those in the literatures. The resonant frequencies with continuous dielectric profiles have also been calculated, which can not be done by using conventional mode-matching methods.

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REFERENCES

Precision Broadband Wavemeter for Millimeter and Submillimeter Range

Y. A. Dryagin, V. V. Parshin, A. F. Krupnov, N. Gopalsami, and A. C. Raptis

Abstract—A precise, broadband, Fabry–Perot wavemeter has been designed and built to measure wavelengths in the millimeter and submillimeter range. The design of the wavemeter is novel in that it enhances the fundamental mode over a wide band and permits determination of the exact longitudinal index of the mode. With the use of an exact mode number in wavelength calculations, high measurement accuracies, to the extent permissible by the quality factor of the resonator, can be obtained. The wavemeter was tested by measuring well-known spectral lines of the OCS molecule in the frequency range of 72-607 GHz. Measurement of 24 OCS lines demonstrated an accuracy of better than 2 x 10^-5 in relative units and 6.87 x 10^-5 in rms units for frequency/wavelength. A discussion of further development and automation of the wavemeter is included.

I. INTRODUCTION

In short-wave millimeter and submillimeter regions, open resonators of the Fabry–Perot type are analogs to closed cavities of the centimeter and millimeter wave regions [1]. They are based on concepts associated with optical frequencies and so are called quasi-optical Fabry–Perot resonators. The most common resonator employs a curved mirror at one end and a flat mirror at the other end. Stable Gaussian-beam resonances of the TEM00 type can be supported in these open resonators [2]. High quality factors on the order of 10^5 are routinely possible, which enable sharp resonances and high measurement accuracy of resonance locations.

Even so, the conventional method of measuring wavelength leads to diminished accuracy. It consists of tuning the high-Q quasi-optical Fabry–Perot resonator to two consecutive resonances (two consecutive longitudinal modes) and measuring the difference between the corresponding positions of a movable mirror. The difference is equal to half of the wavelength, with necessary diffraction corrections. The procedure is subject to two main sources of error.

1) The measured wavelength is the small difference between the two large distances (on the order of 100 mm) between the mirrors at the qth and (q+1)th longitudinal modes. The relative accuracy of measuring each resonance position is on the order of 1/Q = 10^-5, but the relative accuracy of the difference in position is on the order of q/Q = 10^-3.

2) If the oscillator whose radiation wavelength is to be measured drifts by 10^-5 during the time the resonator is tuned from one mode to another, the error in the wavelength measurement will be q x 10^-3 = 10^-2.

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