

# Logic Synthesis and Verification

Jie-Hong Roland Jiang  
江介宏

Department of Electrical Engineering  
National Taiwan University



Fall 2011

1

# Multi-Level Logic Minimization

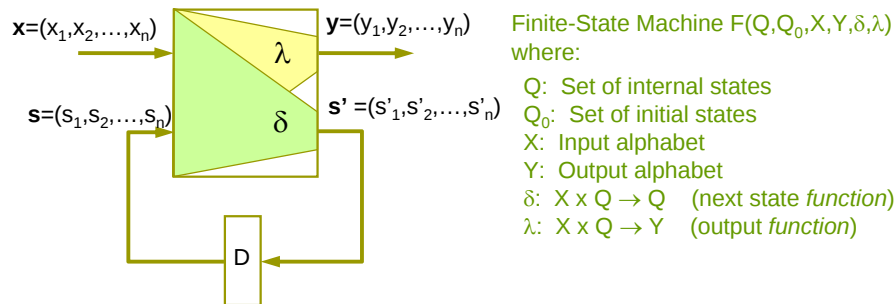
Reading:

*Logic Synthesis in a Nutshell*  
Section 3 (§3.3)

most of the following slides are by  
courtesy of Andreas Kuehlmann

2

## Finite State Machine



Delay element:

- Clocked: synchronous circuit
  - single-phase clock, multiple-phase clocks
- Not clocked: asynchronous circuit

3

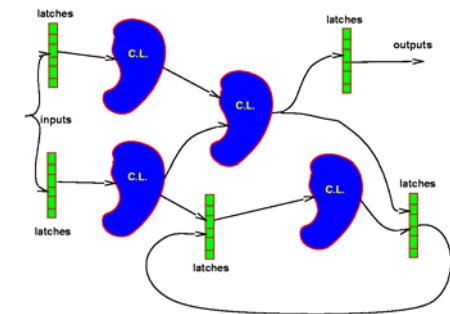
## General Logic Structure

### Combinational optimization

- keep latches/registers at current positions, keep their function
- optimize combinational logic in between

### Sequential optimization

- change latch position/function



4

## Optimization Criteria for Synthesis

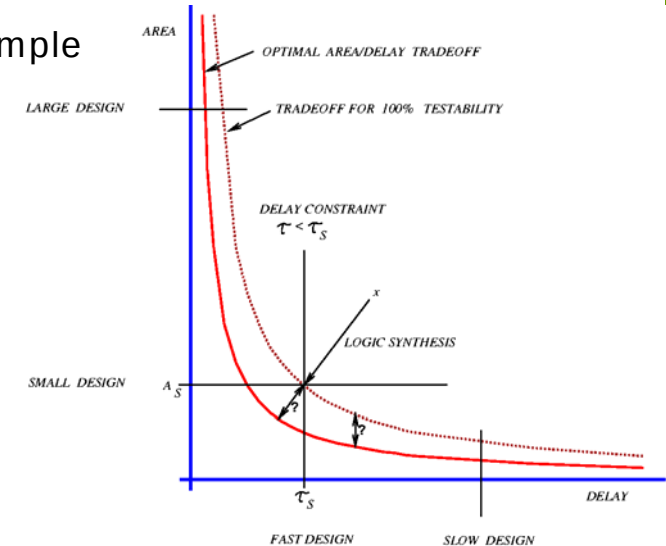
- The optimization criteria for multi-level logic is to **minimize** some function of:
  1. Area occupied by the logic gates and interconnect (approximated by literals = transistors in technology independent optimization)
  2. Critical path delay of the longest path through the logic
  3. Degree of testability of the circuit, measured in terms of the percentage of faults covered by a specified set of test vectors for an approximate fault model (e.g. single or multiple stuck-at faults)
  4. Power consumed by the logic gates
  5. Noise immunity
  6. Placeability, routability

while simultaneously satisfying upper or lower bound constraints placed on these physical quantities

5

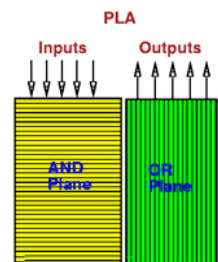
## Area-Delay Trade-off

### Example

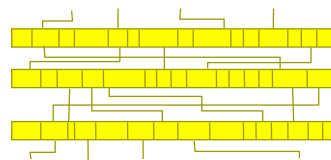


6

## Two-Level (PLA) vs. Multi-Level



E.g. Standard Cell Layout



- PLA
  - Control logic
  - Constrained layout
  - Highly automatic
  - Technology independent
  - Multi-valued logic
  - Input, output, state encoding
  - Predictable

- Multi-level logic
  - Control logic, data path
  - General layout
  - Automatic
  - Partially technology independent
  - Some ideas of multi-valued logic
  - Occasionally involving encoding
  - Hard to predict

7

## General Approaches to Synthesis

### □ PLA synthesis:

- theory well understood
- predictable results in a top-down flow

### □ Multi-level synthesis:

- optimization criteria very complex
  - except special cases, no general theory available
- greedy optimization approach
  - incrementally improve along various dimensions of the criteria
- works on common design representation (circuit or network representation)
  - attempt a change, accept if criteria improve, reject otherwise

8

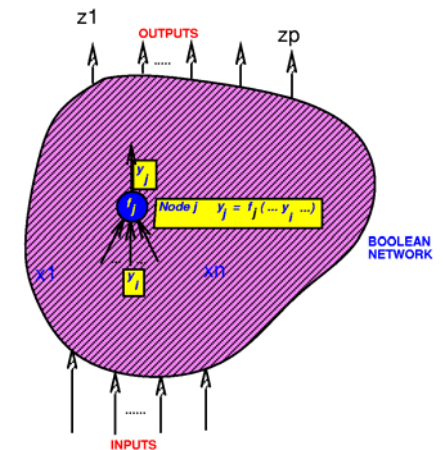
# Transformation-based Synthesis

- All modern synthesis systems are transformation based
  - set of transformations that change network representation
    - work on uniform network representation
  - “script” of “scenario” that can orchestrate various transformations
- Transformations differ in:
  - the scope they are applied
    - Local vs. global restructuring
  - the domain they optimize
    - combinational vs. sequential
    - timing vs. area
    - technology independent vs. technology dependent
  - the underlying algorithms they use
    - BDD based, SAT based, structure based

9

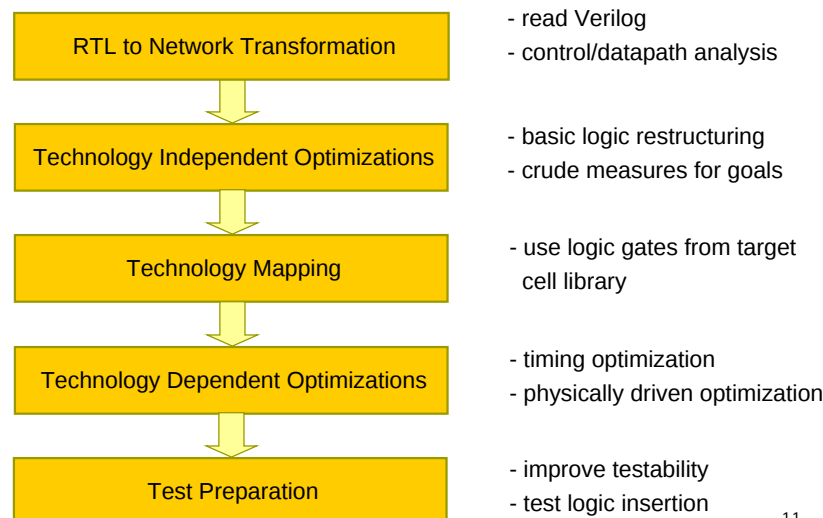
# Network Representation

- Boolean network
  - Directed acyclic graph (DAG)
  - Node logic function representation  $f_j(x,y)$
  - Node variable  $y_j: y_j = f_j(x,y)$
  - Edge  $(i,j)$  if  $f_j$  depends explicitly on  $y_i$
- Inputs:  $x = (x_1, \dots, x_n)$
- Outputs:  $z = (z_1, \dots, z_p)$
- External don't cares:  $d_1(x), \dots, d_p(x)$  for outputs



10

# Typical Synthesis Scenario



11

# Local vs. Global Transformation

- Local transformations optimize one node's function in the network
  - smaller area
  - faster performance
  - map to a particular set of cells
- Global transformations restructure the entire network
  - merging nodes
  - splitting nodes
  - removing/changing connections between nodes
- Node representation:
  - keep size bounded to avoid blow-up of local transformations
    - SOP, POS
    - BDD
    - Factored forms

12

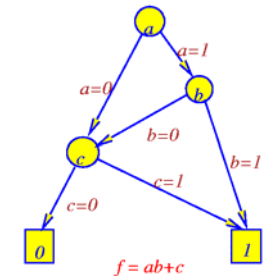
## Sum-of-Products (SOP)

- Example  
 $abc' + a'bd + b'd' + b'e'f$
- Advantages:
  - Easy to manipulate and minimize
  - many algorithms available (e.g. AND, OR, TAUTOLOGY)
  - two-level theory applies
- Disadvantages:
  - Not representative of logic complexity
    - E.g.,  $f = ad + ae + bd + be + cd + ce$  and  $f' = a'b'c' + d'e'$  differ in their implementation by an inverter
  - Not easy to estimate logic; difficult to estimate progress during logic manipulation

13

## Reduced Ordered BDD

- Represents both function and its complement, like factored forms to be discussed
- Like network of muxes, but restricted since controlled by primary input variables
  - not really a good estimator for implementation complexity
- Given an ordering, reduced BDD is canonical, hence a good replacement for truth tables
- For a good ordering, BDDs remain reasonably small for complicated functions (but not multipliers, for instance)
- Manipulations are well defined and efficient
- Only true support variables (dependency on primary input variables) are displayed



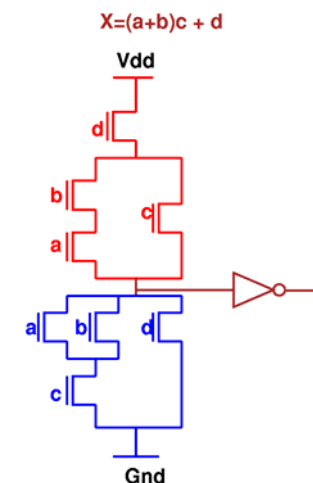
14

## Factor Form

- Example  
 $(ad + b'c)(c + d'(e + ac')) + (d + e)fg$
- Advantages
  - good representative of logic complexity
    - $f = ad + ae + bd + be + cd + ce$
    - $f' = a'b'c' + d'e' \Rightarrow f = (a + b + c)(d + e)$
  - in many designs (e.g. complex gate CMOS) the implementation of a function corresponds directly to its factored form
  - good estimator of logic implementation complexity
  - doesn't blow up easily
- Disadvantages
  - not as many algorithms available for manipulation
  - usually converted into SOP before manipulation

15

## Factor Form



Note:

literal count  $\approx$  transistor count  $\approx$  area

- however, area also depends on wiring, gate size, etc.
- therefore very crude measure

16

## Factored Form

- **Definition:**  $f$  is an **algebraic expression** if  $f$  is a set of cubes (SOP), such that no single cube contains another (minimal with respect to single cube containment)
  - **Example**  
 $a+ab$  is not an algebraic expression (factoring gives  $a(1+b)$  )
- **Definition:** The **product** of two expressions  $f$  and  $g$  is a set defined by  $fg = \{cd \mid c \in f \text{ and } d \in g \text{ and } cd \neq 0\}$ 
  - **Example**  
 $(a+b)(c+d+a') = ac+ad+bc+bd+a'b$
- **Definition:**  $fg$  is an **algebraic product** if  $f$  and  $g$  are algebraic expressions and have **disjoint** support (that is, they have no input variables in common)
  - **Example**  
 $(a+b)(c+d) = ac+ad+bc+bd$  is an algebraic product

17

## Factored Form

- **Definition:** A **factored form** can be defined recursively by the following rules. A factored form is either a product or sum where:
  - a product is either a single **literal** or a **product** of factored forms
  - a sum is either a single **literal** or a **sum** of factored forms
- A **factored form is a parenthesized algebraic expression**
  - In effect a factored form is a **product of sums of products** or a **sum of products of sums**
- **Any logic function** can be represented by a factored form, and **any** factored form is a representation of some logic function

18

## Factored Form

- **Example**
  - $x, y', abc', a+b'c, ((a'+b)cd+e)(a+b')+e'$  are factored forms
  - $(a+b)'c$  is not a factored form since **complement is not allowed, except on literals**
- **Factored forms are not unique**
  - Three equivalent factored forms  
 $ab+c(a+b), \quad bc+a(b+c), \quad ac+b(a+c)$

19

## Factored Form

- **Definition:** The **factorization value** of an algebraic factorization  $F=G_1G_2+R$  is defined to be
 
$$fact\_val(F, G_2) = lits(F) - (lits(G_1) + lits(G_2) + lits(R))$$

$$= (|G_1|-1) lits(G_2) + (|G_2|-1) lits(G_1)$$
  - Assuming  $G_1, G_2$  and  $R$  are algebraic expressions, where  $|H|$  is the number of cubes in the SOP form of  $H$
  - **Example**  
 $F = ae+af+ag+bce+bcf+bcg+bde+bdf+bdg$   
 can be expressed in the form  $F = (a+b(c+d))(e+f+g)$ , which requires 7 literals, rather than 24
  - If  $G_1=(a+bc+bd)$  and  $G_2=(e+f+g)$ , then  $R=\emptyset$  and  $fact\_val(F, G_2) = 2 \times 3 + 2 \times 5 = 16$ 
    - The above factored form saves 17 literals, not 16. The extra literal comes from recursively applying the formula to the factored form of  $G_1$ .

20

## Factored Form

- Factored forms are more **compact** representations of logic functions than the traditional SOP forms

- Example:

$$(a+b)(c+d(e+f(g+h+i+j)))$$

when represented as an SOP form is

$$ac+ade+adfg+adfh+adfi+adfj+bc+bde+bdfg+bdfh+bdfi+bdfj$$

- SOP is a factored form, but it may not be a good factorization

21

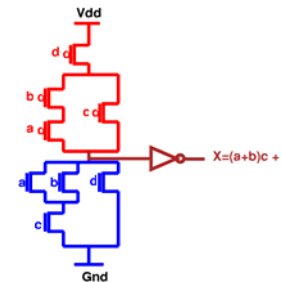
## Factored Form

- There are functions whose size is **exponential** in SOP representation, but **polynomial** in factored form

- Example:

Achilles' heel function  $\prod_{i=1}^{i=n/2} (x_{2i-1} + x_{2i})$

$n$  literals in factored form and  $(n/2) \times 2^{n/2}$  literals in SOP form



Factored forms are useful in **estimating** area and delay in a multi-level synthesis and optimization system. In many design styles (e.g. complex gate CMOS design) the implementation of a function corresponds directly to its factored form.

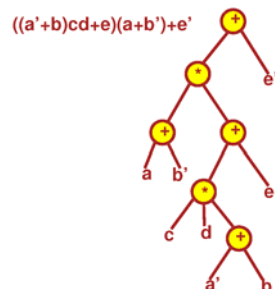
22

## Factored Form

- Factored forms can be graphically represented as labeled **trees**, called factoring trees, in which each internal node including the root is labeled with either  $+$  or  $\times$ , and each leaf has a label of either a variable or its complement

- Example

factoring tree of  $((a'+b)cd+e)(a+b')+e'$



23

## Factored Form

- Definition:** The **size** of a factored form  $F$  (denoted  $\rho(F)$ ) is the number of literals in the factored form

- E.g.,  $\rho((a+b)ca') = 4$ ,  $\rho((a+b+cd)(a'+b')) = 6$

- A factored form of a function is **optimal** if no other factored form has less literals
- A factored form is **positive unate** in  $x$ , if  $x$  appears in  $F$ , but  $x'$  does not. A factored form is **negative unate** in  $x$ , if  $x'$  appears in  $F$ , but  $x$  does not.
- $F$  is **unate** in  $x$  if it is either positive or negative unate in  $x$ , otherwise  $F$  is **binate** in  $x$ 
  - E.g.,  $F = (a+b')c+a'$   
positive unate in  $c$ ; negative unate in  $b$ ; binate in  $a$

24

## Factored Form Cofactor

- The cofactor of a factored form  $F$ , with respect to a literal  $x_1$  (or  $x_1'$ ), is the factored form  $F_{x_1} = F_{x_1=1}(x)$  (or  $F_{x_1'} = F_{x_1=0}(x)$ ) obtained by
  - replacing all occurrences of  $x_1$  by 1, and  $x_1'$  by 0
  - simplifying the factored form using the Boolean algebra identities  
 $1y=y$     $1+y=1$     $0y=0$     $0+y=y$
  - after constant propagation (all constants are removed), part of the factored form may appear as  $G+G$ . In general,  $G$  is in a factored form.

25

## Factored Form Cofactor

- The cofactor of a factored form  $F$ , with respect to a cube  $c$ , is a factored form  $F_c$  obtained by successively cofactoring  $F$  with each literal in  $c$

### ■ Example

$F = (x+y'+z)(x'u+z'y'(v+u'))$  and  $c = vz'$ .

Then

$$F_{z'} = (x+y')(x'u+y'(v+u'))$$

$$F_{z'v} = (x+y')(x'u+y')$$

26

## Factored Form Optimality

### □ Definition

Let  $f$  be a completely specified Boolean function, and  $\rho(f)$  is the minimum number of literals in any factored form of  $f$

- Recall  $\rho(F)$  is the number of literals of a factored form  $F$

### □ Definition

Let  $\text{sup}(f)$  be the true support variable of  $f$ , i.e. the set of variables that  $f$  depends on. Two functions  $f$  and  $g$  are **orthogonal**, denoted  $f \perp g$ , if  $\text{sup}(f) \cap \text{sup}(g) = \emptyset$

27

## Factored Form Optimality

- Lemma: Let  $f = g + h$  such that  $g \perp h$ , then  $\rho(f) = \rho(g) + \rho(h)$

### ■ Proof:

Let  $F$ ,  $G$  and  $H$  be the optimum factored forms of  $f$ ,  $g$  and  $h$ . Since  $G+H$  is a factored form,  $\rho(f) = \rho(F) \leq \rho(G+H) = \rho(g) + \rho(h)$ .

Let  $c$  be a minterm, on  $\text{sup}(g)$ , of  $g'$ . Since  $g$  and  $h$  have disjoint support, we have  $f_c = (g+h)_c = g_c + h_c = 0 + h_c = h_c = h$ . Similarly, if  $d$  is a minterm of  $h'$ ,  $f_d = g$ . Because  $\rho(h) = \rho(f_d) \leq \rho(F_d)$  and  $\rho(g) = \rho(f_c) \leq \rho(F_c)$ ,  $\rho(h) + \rho(g) \leq \rho(F_c) + \rho(F_d)$ .

Let  $m$  ( $n$ ) be the number of literals in  $F$  that are from  $\text{SUPPORT}(g)$  ( $\text{SUPPORT}(h)$ ). When computing  $F_c$  ( $F_d$ ), we replace all the literals from  $\text{SUPPORT}(g)$  ( $\text{SUPPORT}(h)$ ) by the appropriate values and simplify the factored form by eliminating all the constants and possibly some literals from  $\text{sup}(g)$  ( $\text{sup}(h)$ ) by using the Boolean identities. Hence  $\rho(F_c) \leq n$  and  $\rho(F_d) \leq m$ . Since  $\rho(F) = m+n$ ,  $\rho(F_c) + \rho(F_d) \leq m+n = \rho(F)$ . We have  $\rho(f) \leq \rho(g) + \rho(h) \leq \rho(F_c) + \rho(F_d) \leq \rho(F) \Rightarrow \rho(f) = \rho(g) + \rho(h)$  since  $\rho(f) = \rho(F)$ .

28

## Factored Form Optimality

- Note, the previous result does not imply that **all** minimum literal factored forms of  $f$  are sums of the minimum literal factored forms of  $g$  and  $h$
- **Corollary:** Let  $f = gh$  such that  $g \perp h$ , then  $\rho(f) = \rho(g) + \rho(h)$ 
  - **Proof:**  
Let  $F$  denote the factored form obtained using DeMorgan's law. Then  $\rho(F) = \rho(F)$ , and therefore  $\rho(f) = \rho(f')$ . From the above lemma, we have  $\rho(f) = \rho(f') = \rho(g' + h') = \rho(g') + \rho(h') = \rho(g) + \rho(h)$ .
- **Theorem:** Let  $f = \sum_{i=1}^n \prod_{j=1}^m f_{ij}$  such that  $f_{ij} \perp f_{kl}, \forall i \neq j \text{ or } k \neq l$ , then
 
$$\rho(f) = \sum_{i=1}^n \sum_{j=1}^m \rho(f_{ij})$$
  - **Proof:**  
Use induction on  $m$  and then  $n$ , and the above lemma and corollary.

29

## Factored Form

- SOP forms are used as the internal representation of logic functions in most multi-level logic optimization systems
- **Advantages**
  - good algorithms for manipulating them are available
- **Disadvantages**
  - performance is unpredictable - may accidentally generate a function whose SOP form is too large
  - factoring algorithms have to be used constantly to provide an estimate for the size of the Boolean network, and the time spent on factoring may become significant
- **Possible solution**
  - **avoid** SOP representation by using factored forms as the internal representation
  - still not practical unless we know how to perform logic operations **directly** on factored forms without converting to SOP forms
  - the most common logic operations over factored form have been partially provided

30

## Boolean Network Manipulation

- **Basic techniques**
  - **Structural operations (change topology)**
    - Algebraic
    - Boolean
  - **Node simplification (change node functions)**
    - Node minimization using don't cares

31

## Structural Operation

- **Restructuring:** Given initial network, find **best** network
  - **Example**

$$f_1 = abcd + ab'cd' + acd'e + ab'c'd' + a'c + cdf + abc'd'e' + ab'c'df'$$

$$f_2 = bdg + b'dfg + b'd'g + bd'eg$$

minimizing

$$f_1 = bcd + b'cd' + cd'e + a'c + cdf + abc'd'e' + ab'c'df'$$

$$f_2 = bdg + dfg + b'd'g + d'eg$$

factoring

$$f_1 = c(d(b+f) + d'(b'+e) + a') + ac'(bd'e' + b'df')$$

$$f_2 = g(d(b+f) + d'(b'+e))$$

decompose

$$f_1 = c(x+a') + ac'x'$$

$$f_2 = gx$$

$$x = d(b+f) + d'(b'+e)$$
- **Two problems:**
  - find good **common** subfunctions
  - effect the **division**

32

## Structural Operation

### Basic Operations:

- Decomposition (single function)  
 $f = abc + abd + a'c'd' + b'c'd' \Rightarrow$   
 $f = xy + x'y' \quad x = ab \quad y = c+d$
- Extraction (multiple functions)  
 $f = (az + bz')cd + e \quad g = (az + bz')e' \quad h = cde \Rightarrow$   
 $f = xy + e \quad g = xe' \quad h = ye \quad x = az + bz' \quad y = cd$
- Factoring (series-parallel decomposition)  
 $f = ac + ad + bc + bd + e \Rightarrow$   
 $f = (a+b)(c+d) + e$
- Substitution  
 $g = a+b \quad f = a+bc \Rightarrow$   
 $f = g(a+c)$
- Collapsing (also called elimination)  
 $f = ga + g'b \quad g = c+d \Rightarrow$   
 $f = ac + ad + bc'd' \quad g = c+d$

"Division" plays a key role in all of these operations

33

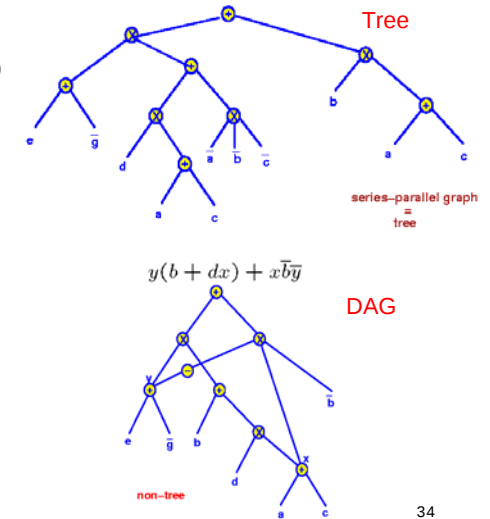
## Factoring vs. Decomposition

### Factoring:

- $f = (e+g')(d(a+c) + a'b'c') + b(a+c)$

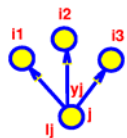
### Decomposition:

- $y(b+dx) + xb'y'$
- Similar to merging common nodes and using negative pointers in BDD. However, **not** canonical, so have no perfect identification of common nodes.



34

## Structural Operation Node Elimination



$$value(j) = \left( \sum_{i \in FO(j)} n_i \right) (l_j - 1) - l_j$$

where

$n_i$  = number of times literals  $y_j$  and  $y_j'$  occur in factored form  $f_i$

■ can treat  $y_j$  and  $y_j'$  the same since  $\rho(F_j) = \rho(F_j')$

$l_j$  = number of literals in factored  $f_j$

with factoring

$$l_j + \sum_{i \in FO(j)} n_i + c$$

without factoring

$$l_j \sum_{i \in FO(j)} n_i + c$$

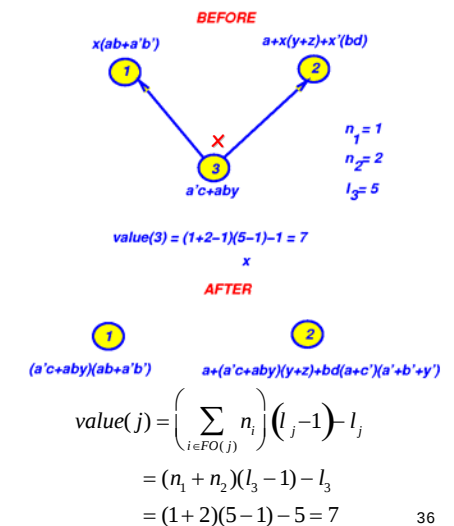
value = (without factoring) - (with factoring)

35

## Structural Operation Node Elimination

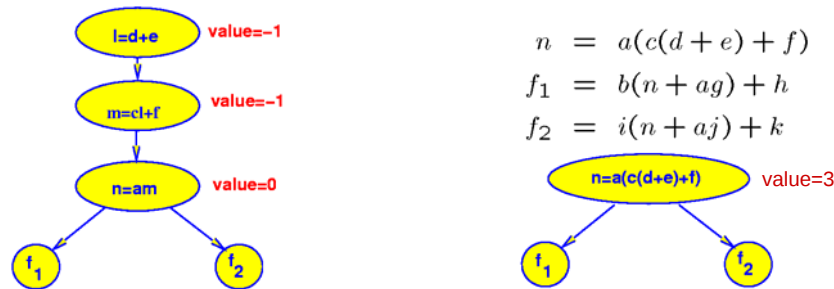
### Example

- Literals before  
 $5 + 7 + 5 = 17$
- Literals after  
 $9 + 15 = 24$
- Difference:  
 after - before =  
 value = 7



36

## Structural Operation Node Elimination



Note: Value of a node can change during elimination

37

## Factorization

Given a SOP, how do we generate a “good” factored form

Division operation:

- is central in many operations
- find a good divisor
- apply division
  - results in quotient and remainder

Applications:

- factoring
- decomposition
- substitution
- extraction

38

## Division

**Definition:** An operation **op** is called **division** if, given two SOP expressions  $F$  and  $G$ , it generates expressions  $H$  and  $R$  ( $\langle H, R \rangle = \text{op}(F, G)$ ) such that  $F = GH + R$

- $G$  is called the divisor
- $H$  is called the quotient
- $R$  is called the remainder

**Definition:** If  $GH$  is an algebraic product, then **op** is called an algebraic division (denoted  $F // G$ ), otherwise  $GH$  is a Boolean product and **op** is called a Boolean division (denoted  $F \div G$ )

39

## Division

Example:

$$f = ad + ae + bcd + j$$

$$g_1 = a + bc$$

$$g_2 = a + b$$

Algebraic division:

$$\square f // a = d + e, r = bcd + j$$

Also,  $f // a = d$  or  $f // a = e$ , i.e. algebraic division is not unique

$$\square f // (bc) = d, r = ad + ae + j$$

$$\square h_1 = f // g_1 = d, r_1 = ae + j$$

Boolean division:

$$\square h_2 = f \div g_2 = (a + c)d, r_2 = ae + j.$$

i.e.  $f = (a+b)(a+c)d + ae + j$

40

## Division

### Definition:

G is an **algebraic factor** of F if there exists an algebraic expression H such that  $F = GH$  (using algebraic multiplication)

### Definition:

G is an **Boolean factor** of F if there exists an expression H such that  $F = GH$  (using Boolean multiplication)

### Example

- $f = ac + ad + bc + bd$ 
  - $(a+b)$  is an algebraic factor of f since  $f = (a+b)(c+d)$
- $f = -ab + ac + bc$ 
  - $(a+b)$  is a Boolean factor of f since  $f = (a+b)(-a+c)$

41

## Why Algebraic Methods?

### Algebraic methods provide fast algorithms for various operations

- Treat logic functions as polynomials
- Fast algorithms for polynomials exist
- Lost of optimality but results are still good
- Can iterate and interleave with Boolean operations
  - In specific instances, slight extensions are available to include Boolean methods

42

## Weak Division

### Weak division is a specific example of algebraic division

### Definition:

Given two algebraic expressions F and G, a division is called a **weak division** if

1. it is algebraic and
  2. remainder R has as few cubes as possible
- The quotient H resulting from weak division is denoted by  $F/G$

### Theorem:

Given expressions F and G, H and R generated by weak division are unique

43

## Weak Division

```
ALGORITHM WEAK_DIV(F,G) {
  // G = {g1, g2, ...}, F = {f1, f2, ...} are sets of cubes
  foreach gi {
    Vgi = ∅
    foreach fj {
      if(fj contains all literals of gi) {
        vij = fj - literals of gi
        Vgi = Vgi ∪ vij
      }
    }
  }
  H = ∩i Vgi
  R = F - GH
  return (H,R);
}
```

44

## Weak Division

### Example

$$F = ace + ade + bc + bd + be + a'b + ab$$

$$G = ae + b$$

$$V^{ae} = c + d$$

$$V^b = c + d + e + a' + a$$

$$H = c + d = F/G$$

$$H = \cap V^{g_i}$$

$$R = be + a'b + ab$$

$$R = F \setminus GH$$

$$F = (ae + b)(c + d) + be + a'b + ab$$

45

## Weak Division

### We use filters to prevent trying a division

#### G is not an algebraic divisor of F if

- G contains a literal not in F,

- G has more terms than F,

- For any literal, its count in G exceeds that in F, or

- F is in the transitive fanin of G.

46

## Weak Division

### Weak\_Div provides a method to divide an expression for a given divisor

### How do we find a “good” divisor?

- Restrict to algebraic divisors
- Generalize to Boolean divisors

### Problem:

Given a set of functions  $\{F_i\}$ , find common weak (algebraic) divisors

47

## Divisor Identification Primary Divisor

### Definition:

An expression is **cube-free** if no cube divides the expression evenly (i.e., there is no literal that is common to all the cubes)

“ab+c” is cube-free

“ab+ac” and “abc” are not cube-free

- Note:** A cube-free expression **must** have more than one cube

### Definition:

The **primary divisors** of an expression F are the set of expressions

$$D(F) = \{F/c \mid c \text{ is a cube}\}$$

Note that  $F/c$  is the **quotient of a weak division**

48

## Divisor Identification Kernel and Co-Kernel

### Definition:

The **kernels** of an expression F are the set of expressions

$$K(F) = \{G \mid G \in D(F) \text{ and } G \text{ is cube-free}\}$$

- In other words, the kernels of an expression F are the **cube-free primary divisors** of F

### Definition:

A cube c used to obtain the kernel  $K = F/c$  is called a **co-kernel** of K

- $C(F)$  is used to denote the **set of co-kernels** of F

49

## Divisor Identification Kernel and Co-Kernel

### Example

$$\begin{aligned} x &= adf + aef + bdf + bef + cdf + cef + g \\ &= (a + b + c)(d + e)f + g \end{aligned}$$

**kernels**

a+b+c

d+e

(a+b+c)(d+e)f+g

**co-kernels**

df, ef

af, bf, cf

1

50

## Divisor Identification Kernel and Kernel Intersection

### Fundamental Theorem

If two expressions F and G have the property that

$$\forall k_F \in K(F), \forall k_G \in K(G) \rightarrow |k_G \cap k_F| \leq 1$$

( $k_G$  and  $k_F$  have at most one term in common),

then F and G have **no common** algebraic divisors with **more than one cube**

#### Important:

If we “kernel” all functions and there are no nontrivial intersections, then the only common algebraic divisors left are **single cube divisors**

51

## Divisor Identification Kernel Level

### Definition:

A kernel is of **level 0** ( $K^0$ ) if it contains no kernels except itself

A kernel is of **level n** or less ( $K^n$ ) if it contains at least one kernel of level (n-1) or less, but no kernels (except itself) of level n or greater

- $K^n(F)$  is the set of kernels of level n or less
- $K^0(F) \subset K^1(F) \subset K^2(F) \subset \dots \subset K^n(F) \subset K(F)$
- level-n kernels =  $K^n(F) \setminus K^{n-1}(F)$

### Example:

$$F = (a + b(c + d))(e + g)$$

$$k_1 = a + b(c + d) \in K^1 \notin K^0 \Rightarrow \text{level-1}$$

$$k_2 = c + d \in K^0$$

$$k_3 = e + g \in K^0$$

52

## Divisor Identification Kerneling Algorithm

```

Algorithm KERNEL(j, G) {
  R =  $\emptyset$ 
  if(CUBE_FREE(G)) R = {G}
  for(i=j+1,...,n) {
    if( $l_i$  appears only in one term) continue
    if( $\exists k \leq i, l_k \in$  all cubes of  $G/l_i$ ) continue
    R = R  $\cup$  KERNEL(i, MAKE_CUBE_FREE( $G/l_i$ ))
  }
  return R
}
MAKE_CUBE_FREE(F) removes algebraic cube factor from F

```

## Divisor Identification Kerneling Algorithm

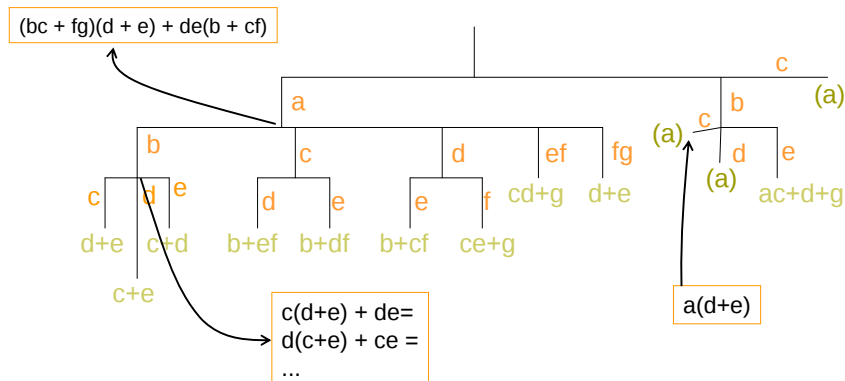
- `KERNEL(0, F)` returns all the kernels of  $F$

□ Note:

- The test “( $\exists k \leq i, I_k \in$  all cubes of  $G/I_i$ )” in the kerneling algorithm is a **major** efficiency factor. It also guarantees that no co-kernel is tried more than once.
- Can be used to generate all co-kernels

## Divisor Identification Kerneling Algorithm

### Example

$$F = abcd + abce + adfg + aefg + adbe + acdef + beg$$


## Divisor Identification Kerneling Algorithm

### Example

| co-kernels | kernels                                  |
|------------|------------------------------------------|
| 1          | $a((bc + fg)(d + e) + de(b + cf)) + beg$ |
| a          | $(bc + fg)(d + e) + de(b + cf)$          |
| ab         | $c(d + e) + de$                          |
| abc        | $d + e$                                  |
| abd        | $c + e$                                  |
| abe        | $c + d$                                  |
| ac         | $b(d + e) + def$                         |
| acd        | $b + ef$                                 |

Note:  $F/bc = ad + ae = a(d + e)$

## Factor

```
Algorithm FACTOR(F) {  
  if(F has no factor) return F  
  // e.g. if |F|=1, or F is an OR of single literals  
  // or of no literal appears more than once  
  D = CHOOSE_DIVISOR(F)  
  (Q,R) = DIVIDE(F,D)  
  return FACTOR(Q)×FACTOR(D) + FACTOR(R) //recur  
}
```

- different heuristics can be applied for CHOOSE\_DIVISOR
- different DIVIDE routines may be applied (algebraic division, Boolean division)

57

## Factor

### □ Example:

$F = abc + abd + ae + af + g$   
 $D = c + d$   
 $Q = ab$   
 $P = ab(c + d) + ae + af + g$   
 $O = ab(c + d) + a(e + f) + g$

Notation:

F = original function  
D = divisor  
Q = quotient  
P = partial factored form  
O = final factored form by  
FACTOR restricting to  
algebraic operations only

### ■ Problem 1:

O is not optimal since not maximally factored and can be further factored to " $a(b(c + d) + e + f) + g$ "

- It occurs when quotient Q is a single cube, and some of the literals of Q also appear in the remainder R

58

## Factor

### □ To solve Problem 1

- Check if the quotient Q is not a single cube, then done

- Else, pick a literal  $l_1$  in Q which occurs most frequently in cubes of F. Divide F by  $l_1$  to obtain a new divisor  $D_1$ .

Now, F has a new partial factored form

$$(l_1)(D_1) + (R_1)$$

and literal  $l_1$  does not appear in  $R_1$ .

- **Note:** The new divisor  $D_1$  contains the original D as a divisor because  $l_1$  is a literal of Q. When recursively factoring  $D_1$ , D can be discovered again.

59

## Factor

### □ Example:

$F = ace + ade + bce + bde + cf + df$   
 $D = a + b$   
 $Q = ce + de$   
 $P = (ce + de)(a + b) + (c + d)f$   
 $O = e(c + d)(a + b) + (c + d)f$

Notation:

F = original function  
D = divisor  
Q = quotient  
P = partial factored form  
O = final factored form by  
FACTOR restricting to  
algebraic operations only

### ■ Problem 2:

O is not maximally factored because " $(c + d)$ " is common to both products " $e(c + d)(a + b)$ " and " $(c + d)f$ "

- The final factored form should have been " $(c+d)(e(a + b) + f)$ "

60

## Factor

### □ To solve Problem 2

#### ■ Essentially, we reverse D and Q!!

- Make Q **cube-free** to get  $Q_1$
- Obtain a new divisor  $D_1$  by dividing F by  $Q_1$
- If  $D_1$  is cube-free, the partial factored form is  $F = (Q_1)(D_1) + R_1$ , and can recursively factor  $Q_1$ ,  $D_1$ , and  $R_1$
- If  $D_1$  is not cube-free, let  $D_1 = cD_2$  and  $D_3 = Q_1D_2$ . We have the partial factoring  $F = cD_3 + R_1$ . Now recursively factor  $D_3$  and  $R_1$ .

61

## Factor

```
Algorithm GFACTOR(F, DIVISOR, DIVIDE) { // good factor
  D = DIVISOR(F)
  if(D = 0) return F
  Q = DIVIDE(F,D)
  if (|Q| = 1) return LF(F, Q, DIVISOR, DIVIDE)
  Q = MAKE_CUBE_FREE(Q)
  (D, R) = DIVIDE(F,Q)
  if (CUBE_FREE(D)) {
    Q = GFACTOR(Q, DIVISOR, DIVIDE)
    D = GFACTOR(D, DIVISOR, DIVIDE)
    R = GFACTOR(R, DIVISOR, DIVIDE)
    return Q x D + R
  }
  else {
    C = COMMON_CUBE(D) // common cube factor
    return LF(F, C, DIVISOR, DIVIDE)
  }
}
```

62

## Factor

```
Algorithm LF(F, C, DIVISOR, DIVIDE) { // literal
  factor
  L = BEST_LITERAL(F, C) //L ∈ C most frequent in F
  (Q, R) = DIVIDE(F, L)
  C = COMMON_CUBE(Q) // largest one
  Q = CUBE_FREE(Q)
  Q = GFACTOR(Q, DIVISOR, DIVIDE)
  R = GFACTOR(R, DIVISOR, DIVIDE)
  return L x C x Q + R
}
```

63

## Factor

- Various kinds of factoring can be obtained by choosing different forms of **DIVISOR** and **DIVIDE**
- **CHOOSE\_DIVISOR**:
  - LITERAL** - chooses most frequent literal
  - QUICK\_DIVISOR** - chooses the first level-0 kernel
  - BEST\_DIVISOR** - chooses the best kernel
- **DIVIDE**:
  - Algebraic Division
  - Boolean Division

64

## Factor

### Example

$$x = ac + ad + ae + ag + bc + bd + be + bf + ce + cf + df + dg$$

#### LITERAL\_FACTOR:

$$x = a(c + d + e + g) + b(c + d + e + f) + c(e + f) + d(f + g)$$

#### QUICK\_FACTOR:

$$x = g(a + d) + (a + b)(c + d + e) + c(e + f) + f(b + d)$$

#### GOOD\_FACTOR:

$$(c + d + e)(a + b) + f(b + c + d) + g(a + d) + ce$$

65

## Factor

- QUICK\_FACTOR uses GFACTOR, first level-0 kernel DIVISOR, and WEAK\_DIV

### Example

$$x = ae + afg + afh + bce + bcfg + bcfh + bde + bdfg + bcfh$$

$$D = c + d \quad \text{---- level-0 kernel (first found)}$$

$$Q = x/D = b(e + f(g + h)) \quad \text{---- weak division}$$

$$Q = e + f(g + h) \quad \text{---- make cube-free}$$

$$(D, R) = \text{WEAK\_DIV}(x, Q) \quad \text{---- second division}$$

$$D = a + b(c + d)$$

$$x = QD + R, \quad R = 0$$

$$x = (e + f(g + h)) (a + b(c + d))$$

66

## Decomposition

- Decomposition is the same as factoring **except**:

- divisors are added as **new** nodes in the network
- the new nodes may **fan out** elsewhere in the network in both **positive** and **negative** phases

```

Algorithm DECOMP( $f_i$ ) {
   $k = \text{CHOOSE\_KERNEL}(f_i)$ 
  if ( $k == 0$ ) return
   $f_{m+j} = k$  // create new node  $m + j$ 
   $f_i = (f_i/k)y_{m+j} + (f_i/k')y'_{m+j} + r$  // change node  $i$  using
  // new node for kernel
  DECOMP( $f_i$ )
  DECOMP( $f_{m+j}$ )
}

```

Similar to factoring, we can define

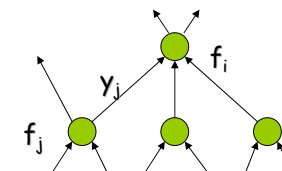
QUICK\_DECOMP: pick a level 0 kernel and improve it

GOOD\_DECOMP: pick the best kernel

67

## Substitution

- Idea:** An existing node in a network may be a useful divisor in another node. If so, no loss in using it (unless delay is a factor).
- Algebraic substitution consists of the process of algebraically dividing the function  $f_i$  at node  $i$  in the network by the function  $f_j$  (or by  $f'_j$ ) at node  $j$ . During substitution, if  $f_j$  is an algebraic divisor of  $f_i$ , then  $f_i$  is transformed into  $f_i = qy_j + r$  (or  $f_i = q_1y_j + q_0y'_j + r$ )
- In practice, this is tried for each node pair of the network.  $n$  nodes in the network  $\Rightarrow O(n^2)$  divisions.



68

## Extraction

- Recall: Extraction operation identifies common sub-expressions and restructures a Boolean network
  - Combine decomposition and substitution to provide an effective extraction algorithm

Algorithm **EXTRACT**

```

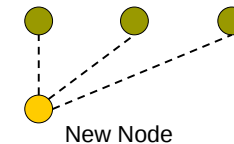
foreach node n {
    DECOMP(n) // decompose all network nodes
}
foreach node n {
    RESUB(n) // resubstitute using existing nodes
}
ELIMINATE_NODES_WITH_SMALL_VALUE
    
```

69

## Extraction

- Kernel Extraction:

1. Find all kernels of all functions
2. Choose kernel intersection with best "value"
3. Create new node with this as function
4. Algebraically substitute new node everywhere
5. Repeat 1,2,3,4 until best value  $\leq$  threshold



70

## Extraction

- Example

$$f_1 = ab(c(d + e) + f + g) + h$$

$$f_2 = ai(c(d + e) + f + j) + k$$

(only level-0 kernels used in this example)

1. Extraction:

$$K^0(f_1) = K^0(f_2) = \{d + e\}$$

$$K^0(f_1) \cap K^0(f_2) = \{d + e\}$$

$$l = d + e$$

$$f_1 = ab(cl + f + g) + h$$

$$f_2 = ai(cl + f + j) + k$$

2. Extraction:

$$K^0(f_1) = \{cl + f + g\}; K^0(f_2) = \{cl + f + j\}$$

$$K^0(f_1) \cap K^0(f_2) = cl + f$$

$$m = cl + f$$

$$f_1 = ab(m + g) + h$$

$$f_2 = ai(m + j) + k$$

No kernel intersections anymore!!

3. Cube extraction:

$$n = am$$

$$f_1 = b(n + ag) + h$$

$$f_2 = i(n + aj) + k$$

71

## Extraction Rectangle Covering

- Alternative method for extraction

- Build co-kernel cube matrix  $M = R^T C$

- rows correspond to co-kernels of individual functions
- columns correspond to individual cubes of kernel
- $m_{ij}$  = cubes of functions
- $m_{ij} = 0$  if cube not there

- Rectangle covering:

- identify sub-matrix  $M^* = R^{*T} C^*$ , where  $R^* \subseteq R$ ,  $C^* \subseteq C$ , and  $m^*_{ij} \neq 0$
- construct divisor  $d$  corresponding to  $M^*$  as new node
- extract  $d$  from all functions

72

## Extraction Rectangle Covering

### Example

$$F = af + bf + ag + cg + ade + bde + cde$$

$$G = af + bf + ace + bce$$

$$H = ade + cde$$

Kernels/Co-kernels:

$$F: (de+fg)/a$$

$$(de+f)/b$$

$$(a+b+c)/de$$

$$(a+b)/f$$

$$(de+g)/c$$

$$(a+c)/g$$

$$G: (ce+f)/\{a,b\}$$

$$(a+b)/\{f,ce\}$$

$$H: (a+c)/de$$

|       |    | a   | b   | c   | ce  | de  | f  | g  |
|-------|----|-----|-----|-----|-----|-----|----|----|
| F     | a  |     |     |     |     | ade | af | ag |
| F     | b  |     |     |     |     | bde | bf |    |
| F     | de | ade | bde | cde |     |     |    |    |
| F     | f  | af  | bf  |     |     |     |    |    |
| M = F | c  |     |     |     |     | cde |    | cg |
| F     | g  | ag  |     | cg  |     |     |    |    |
| G     | a  |     |     |     | ace |     | af |    |
| G     | b  |     |     |     | bce |     | bf |    |
| G     | ce | ace | bce |     |     |     |    |    |
| G     | f  | af  | bf  |     |     |     |    |    |
| H     | de | ade |     | cde |     |     |    |    |

73

## Extraction Rectangle Covering

### Example (cont'd)

$$F = af + bf + ag + cg + ade + bde + cde$$

$$G = af + bf + ace + bce$$

$$H = ade + cde$$

Pick sub-matrix M'

Extract new expression X

$$F = fx + ag + cg + dex + cde$$

$$G = fx + cex$$

$$H = ade + cde$$

$$X = a + b$$

Update M

|       |    | a   | b   | c   | ce  | de  | f  | g  |
|-------|----|-----|-----|-----|-----|-----|----|----|
| F     | a  |     |     |     |     | ade | af | ag |
| F     | b  |     |     |     |     | bde | bf |    |
| F     | de | ade | bde | cde |     |     |    |    |
| F     | f  | af  | bf  |     |     |     |    |    |
| M = F | c  |     |     |     |     | cde |    | cg |
| F     | g  | ag  |     | cg  |     |     |    |    |
| G     | a  |     |     |     | ace |     | af |    |
| G     | b  |     |     |     | bce |     | bf |    |
| G     | ce | ace | bce |     |     |     |    |    |
| G     | f  | af  | bf  |     |     |     |    |    |
| H     | de | ade |     | cde |     |     |    |    |

74

## Extraction Rectangle Covering

### Number literals before - Number of literals after

$$V(R', C') = \sum_{i \in R, j \in C} v_{ij} - \sum_{i \in R'} w_i^r - \sum_{j \in C'} w_j^c$$

$v_{ij}$ : Number of literals of cube  $m_{ij}$

$w_i^r$ : (Number of literals of the cube associated with row  $i$ ) + 1

$w_j^c$ : Number of literals of the cube associated with column  $j$

For prior example

$$V = 20 - 10 - 2 = 8$$

|       |    | a   | b   | c   | ce  | de  | f  | g  |
|-------|----|-----|-----|-----|-----|-----|----|----|
| F     | a  |     |     |     |     | ade | af | ag |
| F     | b  |     |     |     |     | bde | bf |    |
| F     | de | ade | bde | cde |     |     |    |    |
| F     | f  | af  | bf  |     |     |     |    |    |
| M = F | c  |     |     |     |     | cde |    | cg |
| F     | g  | ag  |     | cg  |     |     |    |    |
| G     | a  |     |     |     | ace |     | af |    |
| G     | b  |     |     |     | bce |     | bf |    |
| G     | ce | ace | bce |     |     |     |    |    |
| G     | f  | af  | bf  |     |     |     |    |    |
| H     | de | ade |     | cde |     |     |    |    |

75

## Extraction Rectangle Covering

### Pseudo Boolean Division

Idea: consider entries in covering matrix that are don't cares

overlap of rectangles ( $a+a = a$ )

product that cancel each other out ( $a \cdot a' = 0$ )

Example:

$$F = ab' + ac' + a'b + a'c + bc' + b'$$

Result:

$$X = a' + b' + c'$$

$$F = ax + bx + cx$$

|       |    | a   | b   | c   | a'  | b'  | c'  |
|-------|----|-----|-----|-----|-----|-----|-----|
| F     | a  |     |     |     | *   | ab' | ac' |
| F     | b  |     |     |     | a'b | *   | bc' |
| M = F | c  |     |     |     | a'c | b'c | *   |
| F     | a' | *   | a'b | a'c |     |     |     |
| F     | b' | ab' | *   | b'c |     |     |     |
| F     | c' | ac' | bc' | *   |     |     |     |

76

## Fast Kernel Computation

- Non-robustness of kernel extraction
  - Recomputation of kernels after every substitution: expensive
  - Some functions may have many kernels (e.g. symmetric functions)
- Cannot measure if kernel can be used as complemented node
- Solution: compute only subset of kernels:
  - Two-cube “kernel” extraction [Rajski et al '90]
  - Objects:
    - 2-cube divisors
    - 2-literal cube divisors
  - Example:  $f = abd + a'b'd + a'cd$ 
    - $ab + a'b'$ ,  $b' + c$  and  $ab + a'c$  are 2-cube divisors.
    - $a'd$  is a 2-literal cube divisor.

77

## Fast Kernel Computation

- Properties of fast divisor (kernel) extraction:
  - $O(n^2)$  number of 2-cube divisors in an  $n$ -cube Boolean expression
  - Concurrent extraction of 2-cube divisors and 2-literal cube divisors
  - Handle divisor and complemented divisor simultaneously
- Example:
 
$$f = abd + a'b'd + a'cd$$

$$k = ab + a'b', \quad k' = ab' + a'b \quad (\text{both 2-cube divisors})$$

$$j = ab + a'c, \quad j' = ab' + a'c' \quad (\text{both 2-cube divisors})$$

$$c = ab \quad (2\text{-literal cube}), \quad c' = a' + b' \quad (2\text{-cube divisor})$$

78

## Fast Kernel Computation

- Generating all two cube divisors
 
$$F = \{c_i\}$$

$$D(F) = \{d \mid d = \text{make\_cube\_free}(c_i + c_j)\}$$
  - $c_i, c_j$  are any pair of cubes in  $F$ 
    - I.e., take all pairs of cubes in  $F$  and makes them cube-free
  - Divisor generation is  $O(n^2)$ , where  $n$  = number of cubes in  $F$
- Example:
 
$$F = axe + ag + bcxe + bcg$$

$$\text{make\_cube\_free}(c_i + c_j) = \{xe + g, a + bc, axe + bcg, ag + bcxe\}$$
  - Note: Function  $F$  is made into an algebraic expression before generating double-cube divisors
  - Not all 2-cube divisors are kernels (why?)

79

## Fast Kernel Computation

- Key results of 2-cube divisors

**Theorem:** Expressions  $F$  and  $G$  have a common multiple-cube divisors **if and only if**  $D(F) \cap D(G) \neq \emptyset$

**Proof:**

**If:**

If  $D(F) \cap D(G) \neq \emptyset$  then  $\exists d \in D(F) \cap D(G)$  which is a double-cube divisor of  $F$  and  $G$ .  $d$  is a multiple-cube divisor of  $F$  and of  $G$ .

**Only if:**

Suppose  $C = \{c_1, c_2, \dots, c_m\}$  is a multiple-cube divisor of  $F$  and of  $G$ . Take any  $e = (c_i + c_j)$ . If  $e$  is cube-free, then  $e \in D(F) \cap D(G)$ . If  $e$  is not cube-free, then let  $d = \text{make\_cube\_free}(c_i + c_j)$ .  $d$  has 2 cubes since  $F$  and  $G$  are algebraic expressions. Hence  $d \in D(F) \cap D(G)$ .

80

## Fast Kernel Computation

### Example:

Suppose that  $C = ab + ac + f$  is a multiple divisor of  $F$  and  $G$

If  $e = ac + f$ ,  $e$  is cube-free and  $e \in D(F) \cap D(G)$

If  $e = ab + ac$ ,  $d = \{b + c\} \in D(F) \cap D(G)$

As a result of the Theorem, all multiple-cube divisors can be “discovered” by using just double-cube divisors

81

## Fast Kernel Computation

### Algorithm:

- Generate and store all 2-cube kernels (2-literal cubes) and recognize complement divisors
- Find the best 2-cube kernel or 2-literal cube divisor at each stage and extract it
- Update 2-cube divisor (2-literal cubes) set after extraction
- Iterate extraction of divisors until no more improvement

### Results:

- Much faster
- Quality as good as that of kernel extraction

82

## Boolean Division

### What's wrong with algebraic division?

- Divisor and quotient are orthogonal!
- Better factored form might be:
  - $(g_1 + g_2 + \dots + g_n) (d_1 + d_2 + \dots + d_m)$ 
    - $g_i$  and  $d_j$  may share same literals
    - redundant product literals
      - Example
 
$$abe+ace+abd+cd / (ae+d) = \emptyset$$

$$\text{But: } aabe+ace+abd+cd / (ae+d) = (ab+c)$$
    - $g_i$  and  $d_j$  may share opposite literals
    - product terms are non-existing
      - Example
 
$$a'b+ac+bc / (a'+c) = \emptyset$$

$$\text{But: } a'a+a'b+ac+bc / (a'+c) = (a+b)$$

83

## Boolean Division

### Definition:

$g$  is a **Boolean divisor** of  $f$  if  $h$  and  $r$  exist such that  $f = gh + r$ ,  $gh \neq 0$

$g$  is said to be a **factor** of  $f$  if, in addition,  $r = 0$ , i.e.,  $f = gh$

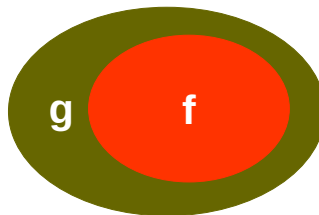
- $h$  is called the **quotient**
- $r$  is called the **remainder**
- $h$  and  $r$  may **not** be unique

84

## Boolean Division

### □ Theorem:

A logic function  $g$  is a **Boolean factor** of a logic function  $f$  if and only if  $f \subseteq g$  (i.e.  $fg' = 0$ , i.e.  $g' \subseteq f'$ )



85

## Boolean Division

### Proof:

$(\Rightarrow)$   $g$  is a Boolean factor of  $f$ . Then  $\exists h$  such that  $f = gh$ ;  
Hence,  $f \subseteq g$  (as well as  $h$ ).

$(\Leftarrow)$   $f \subseteq g \Rightarrow f = gf = g(f + r) = gh$ . (Here  $r$  is any function  $r \subseteq g'$ .)

### □ Note:

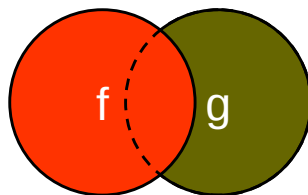
- $h = f$  works fine for the proof
- Given  $f$  and  $g$ ,  $h$  is not unique
- To get a small  $h$  is the same as to get a small  $f + r$ . Since  $rg = 0$ , this is the same as minimizing (simplifying)  $f$  with  $DC = g'$ .

86

## Boolean Division

### □ Theorem:

$g$  is a Boolean divisor of  $f$  if and only if  $fg \neq 0$



87

## Boolean Division

### Proof:

$(\Rightarrow)$   $f = gh + r$ ,  $gh \neq 0 \Rightarrow fg = gh + gr$ . Since  $gh \neq 0$ ,  $fg \neq 0$ .

$(\Leftarrow)$  Assume that  $fg \neq 0$ .  $f = fg + fg' = g(f + k) + fg'$ . (Here  $k \subseteq g'$ .)

Then  $f = gh + r$ , with  $h = f + k$ ,  $r = fg'$ . Since  $gh = fg \neq 0$ , then  $gh \neq 0$ .

### □ Note:

- $f$  has many divisors. We are looking for some  $g$  such that  $f = gh + r$ , where  $g, h, r$  are simple functions. (simplify  $f$  with  $DC = g'$ )

88

## Boolean Division Incomplete Specified Function

□  $F = (f, d, r)$

□ Definition:

A completely specified logic function  $g$  is a **Boolean divisor of  $F$**  if there exist  $h, e$  (completely specified) such that

$$f \subseteq gh + e \subseteq f + d$$

and  $gh \not\subseteq d$ .

□ Definition:

$g$  is a **Boolean factor** of  $F$  if there exists  $h$  such that

$$f \subseteq gh \subseteq f + d$$

89

## Boolean Division Incomplete Specified Function

□ Lemma:

$f \subseteq g$  if and only if  $g$  is a Boolean factor of  $F$ .

Proof:

( $\Rightarrow$ ) Assume that  $f \subseteq g$ . Let  $h = f + k$  where  $kg \subseteq d$ .

Then  $hg = (f + k)g \subseteq (f + d)$ .

Since  $f \subseteq g$ ,  $fg = f$  and thus  $f \subseteq (f + k)g = gh$ .

Thus

$$f \subseteq (f + k)g \subseteq f + d$$

( $\Leftarrow$ ) Assume that  $f = gh$ .

Suppose  $\exists$  minterm  $m$  such that  $f(m) = 1$  but  $g(m) = 0$ .

Then  $f(m) = 1$  but  $g(m)h(m) = 0$  implying that  $f \not\subseteq gh$ .

Thus  $f(m) = 1$  implies  $g(m) = 1$ , i.e.  $f \subseteq g$

□ Note:

■ Since  $kg \subseteq d$ ,  $k \subseteq (d + g')$ . Hence obtain  $h = f + k$  by simplifying  $f$  with  $DC = (d + g')$ .

90

## Boolean Division Incomplete Specified Function

□ Lemma:

$fg \neq 0$  if and only if  $g$  is a Boolean divisor of  $F$ .

Proof:

( $\Rightarrow$ ) Assume  $fg \neq 0$ .

Let  $fg \subseteq h \subseteq (f + d + g')$  and  $fg' \subseteq e \subseteq (f + d)$ .

Then  $f = fg + fg' \subseteq gh + e \subseteq g(f + d + g') + f + d = f + d$

Also,  $0 \neq fg \subseteq gh \rightarrow ghf \neq 0$ .

Now  $gh \not\subseteq d$ , since otherwise  $ghf = 0$  (since  $fd = 0$ ), verifying the conditions of Boolean division.

( $\Leftarrow$ ) Assume that  $g$  is a Boolean divisor.

Then  $\exists h$  such that  $gh \not\subseteq d$  and

$$f \subseteq gh + e \subseteq f + d$$

Since  $gh = (ghf + gh'd) \not\subseteq d$ , then  $fgh \neq 0$  implying that  $fg \neq 0$ .

91

## Boolean Division Incomplete Specified Function

□ Recipe for Boolean division:

$$(f \subseteq gh + e \subseteq f + d)$$

■ Choose  $g$  such that  $fg \neq 0$

■ Simplify  $fg$  with  $DC = (d + g')$  to get  $h$

■ Simplify  $fg'$  with  $DC = (d + fg)$  to get  $e$  (could use  $DC = d + gh$ )

$$\square fg \subseteq h \subseteq f + d + g'$$

$$fg' \subseteq e \subseteq fg' + d + fg = f + d$$

92

## Boolean Division

- Given  $F = (f, d, r)$ , write a cover for  $F$  in the form  $gh + e$  where  $h$  and  $e$  are minimal in some sense

Algorithm:

- Create a new variable  $x$  to "represent"  $g$
- Form the don't care set ( $\tilde{d} = xg' + x'g$ )  
(Since  $x = g$  we don't care if  $x \neq g$ )
- Minimize  $(f\tilde{d}', d + \tilde{d}, r\tilde{d}')$  to get  $\tilde{f}$
- Return  $(h = \tilde{f}/x, e)$  where  $e$  is the remainder of  $\tilde{f}$   
(These are simply the terms not containing  $x$ )
- $f/x$  denote weak algebraic division

93

## Boolean Division

- Note that  $(f\tilde{d}', d + \tilde{d}, r\tilde{d}')$  is a partition. We can use ESPRESSO to minimize it, but the objective there is to minimize the number of cubes - not completely appropriate.

- Example:  
 $f = a + bc$   
 $g = a + b$

$$\tilde{d} = xa'b' + x'(a+b) \text{ where } x = g = (a+b)$$

- Minimize  $(a + bc)\tilde{d}' = (a + bc)(x'a'b' + x(a+b)) = xa + xbc$  with  $DC = xa'b' + x'(a+b)$
- A minimum cover is  $a + bc$  but it does not use  $x$  or  $x'$ !
- Force  $x$  in the cover. This yields  $f = a + xc = a + (a+b)c$ .

Heuristic:

Find answer with  $x$  in it and which also uses the least variables (or literals)

94

## Boolean Division

Assume  $F$  is a cover for  $\mathfrak{F} = (f, d, r)$  and  $D$  is a cover for  $d$ .

First Algorithm:

```
Algorithm Boolean_Divide1(F,D,G) {
  D1 = D + xG' + x'G           // (don't care)
  F1 = FD1'                     // (care on-set)
  R1 = (F1 + D1)' = F1'D1' = F'D1' // (care off-set)
  F2 = remove x' from F1        // positive substitution only
  F3 = MIN_LITERAL(F2, R1, x)   // Filter for Espresso
                                // (minimum literal support including x)
  F4 = ESPRESSO(F3, D1, R1)
  H = F4/x                      // (quotient)
  E = F4 - {xH}                 // (remainder)
  return (HG+E)
}
```

95

## Boolean Division

Assume  $F$  is a cover for  $\mathfrak{F} = (f, d, r)$  and  $D$  is a cover for  $d$ .

Second Algorithm:

```
Algorithm Boolean_Divide2(F,D,G) {
  D1 = D + xG' + x'G           // (don't care)
  F1 = FD1'                     // (on-set)
  R1 = (F1 + D1)' = F1'D1' = F'D1' // (off-set)
  // F2 = remove x' from F1 (difference to first alg.)
  F3 = MIN_LITERAL(F2, R1, x, x') // Filter for Espresso
                                // (minimum literal support including x)
  F4 = ESPRESSO(F3, D1, R1)
  H1 = F4/x                     // (first quotient)
  H0 = F4/x'                    // (first quotient)
  E = F4 - ({xH1}+{x'H0})      // (remainder)
  return (GH1+G'H0+E)
}
```

96

## Boolean Division Minimal Literal Support

### Support minimization (MINVAR)

Given:

$$\mathfrak{I} = (f, d, r)$$

$$F = \{c^1, c^2, \dots, c^k\} \quad (\text{a cover of } \mathfrak{I})$$

$$R = \{r^1, r^2, \dots, r^m\} \quad (\text{a cover of } r)$$

1. Construct blocking matrix  $B^i$  for each  $c^i$
2. Form "super" blocking matrix  $B$
3. Find a minimum cover  $S$  of  $B$ ,

$$S = \{j_1, j_2, \dots, j_v\}.$$

4. Modify  $\tilde{F} \leftarrow \{\tilde{c}^1, \tilde{c}^2, \dots, \tilde{c}^k\}$  where

$$(\tilde{c}^i)_j = \begin{cases} (c^i)_j & \text{if } j \in S \\ \{0, 1\} = 2 & \text{otherwise} \end{cases}$$

$$B = \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^k \end{bmatrix}$$

97

## Boolean Division Minimal Literal Support

### Given:

$$\mathfrak{I} = (f, d, r)$$

$$F = \{c^1, c^2, \dots, c^k\} \quad (\text{a cover of } \mathfrak{I})$$

$$R = \{r^1, r^2, \dots, r^m\} \quad (\text{a cover of } r)$$

$n$ : number of variables

Literal Blocking Matrix:

$$(\hat{B}^i)_{qj} = \begin{cases} 1 & \text{if } v_j \in c^i \text{ and } v_j \in r^q \\ 0 & \text{otherwise} \end{cases}$$

$$(\hat{B}^i)_{q, j+n} = \begin{cases} 1 & \text{if } v_j \in c^i \text{ and } v_j \in r^q \\ 0 & \text{otherwise} \end{cases}$$

### Example:

$$c^i = ad'e', r^q = a'ce$$

$$\hat{B}_q^i = \begin{matrix} abcde a'b'c'd'e' \\ 1000000001 \end{matrix}$$

98

## Boolean Division Minimal Literal Support

### Example (literal blocking matrix)

on-set cube:  $c^i = ab'd$

off-set:  $r = a'b'd' + abd' + acd' + bcd + c'd'$

|        | a | b | c | d | a' | b' | c' | d' |
|--------|---|---|---|---|----|----|----|----|
| a'b'd' | 1 | 0 | 0 | 1 | 0  | 0  | 0  | 0  |
| abd'   | 0 | 0 | 0 | 1 | 0  | 1  | 0  | 0  |
| acd'   | 0 | 0 | 0 | 1 | 0  | 0  | 0  | 0  |
| bcd    | 0 | 0 | 0 | 0 | 0  | 1  | 0  | 0  |
| c'd'   | 0 | 0 | 0 | 1 | 0  | 0  | 0  | 0  |

Minimum column cover  $\{d, b'\}$

Thus  $b'd$  is the maximum prime covering  $ab'd$

### Note:

For one cube, minimum literal support is the same as minimum variable support

99

## Boolean Division

### Example

$$F = a + bc$$

Algebraic division:  $F/(a + b) = 0$

Boolean division:  $F \div (a + b) = a + c$

1. Let  $x = a + b$

2. Generate don't care set:  $D_1 = x'(a + b) + xa'b'$ .

3. Generate care on-set:

$$\square F_1 = F \cap D_1' = (a + bc)(xa + xb + x'a'b') = ax + bcx.$$

$$\square \text{ Let } C = \{c^1 = ax, c^2 = bcx\}$$

4. Generate care off-set:

$$\square R_1 = F'D_1' = (a'b' + a'c')(xa + xb + x'a'b') = a'bc'x + a'b'x'.$$

$$\square \text{ Let } R = \{r^1 = a'bc'x, r^2 = a'b'x'\}.$$

5. Form super-variable blocking matrix using column order  $(a, b, c, x)$ , with  $a', b', c', x'$  omitted.

$$B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1001 \\ 0010 \\ 0101 \end{bmatrix}$$

100

# Boolean Division

## □ Example (cont'd)

6. Find minimum column cover = {a, c, x}
7. Eliminate in  $F_1$  all variables associated with b  
So  $F_1 = ax + bcx = ax + cx = x(a + c)$
8. Simplifying (applying expand, irredundant on  $F_1$ ), we get  $F_1 = a + xc$
9. Thus quotient =  $F_1/x = c$ , remainder = a
10.  $F = a + bc = a + cx = a + c(a + b)$

It is important that x is forced in the cover!

$$B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix} = \begin{matrix} & \begin{matrix} a & b & c & x \end{matrix} \\ \begin{bmatrix} 1000 \\ 1001 \\ 0010 \\ 0101 \end{bmatrix} \end{matrix}$$