Logic Synthesis and Verification

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Boolean Algebra

Boolean Algebra

- Reading

  F. M. Brown. *Boolean Reasoning: The Logic of Boolean Equations.* Dover, 2003. (Chapters 1-3)

Boolean Algebra

- Outline
  - Definitions
  - Examples
  - Properties
  - Boolean formulae and Boolean functions

Boolean Algebra

- A Boolean algebra is an algebraic structure ($\mathbf{B}$, +, ·, $\underline{0}$, $\underline{1}$)
  - $\mathbf{B}$ is a set, called the *carrier*
  - + and · are binary operations defined on $\mathbf{B}$
  - $\underline{0}$ and $\underline{1}$ are distinct members of $\mathbf{B}$

  that satisfies the following postulates (axioms):

  1. *Commutative laws*
  2. *Distributive laws*
  3. *Identities*
  4. *Complements*

Postulates of Boolean Algebra

($\mathbf{B}$, +, ·, $\underline{0}$, $\underline{1}$)

1. $\mathbf{B}$ is closed under + and ·
   $\forall a,b \in \mathbf{B}$, $a + b \in \mathbf{B}$ and $a \cdot b \in \mathbf{B}$
2. Commutative laws: $\forall a,b \in \mathbf{B}$
   $a + b = b + a$
   $a \cdot b = b \cdot a$
3. Distributive laws: $\forall a,b \in \mathbf{B}$
   $a + (b \cdot c) = (a + b) \cdot (a + c)$
   $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Identities: $\forall a \in \mathbf{B}$
   $\underline{0} + a = a$
   $\underline{1} \cdot a = a$
5. Complements: $\forall a \in \mathbf{B}$, $\exists\, a' \in \mathbf{B}$ s.t.
   $a + a' = \underline{1}$
   $a \cdot a' = \underline{0}$
   Verify that a' is unique in ($\mathbf{B}$, +, ·, $\underline{0}$, $\underline{1}$).

Instances of Boolean Algebra

- Switching algebra (two-element Boolean algebra)
- The algebra of classes (subsets of a set)
- Arithmetic Boolean algebra
- The algebra of propositional functions

Instance 1: Switching Algebra

- A switching algebra is a two-element Boolean Algebra ($\{0,1\}$, +, ·, 0, 1) consisting of:
  - the set $\mathbf{B} = \{0, 1\}$
  - two binary operations AND(·) and OR(+)
  - one unary operation NOT(′)

  where

OR	0	1
0	0	1
1	1	1

AND	0	1
0	0	0
1	0	1

NOT	-
0	1
1	0

Switching Algebra

- Just one of many other Boolean algebras
  - (Ex: verify that the algebra satisfies all the postulates.)
- An exclusive property (not hold for all Boolean algebras) for two-element Boolean algebra:
  $x + y = 1$ *iff* $x=1$ or $y=1$
  $x \cdot y = 0$ *iff* $x=0$ or $y=0$

OR	0	1
0	0	1
1	1	1

AND	0	1
0	0	0
1	0	1

NOT	-
0	1
1	0

Instance 2: Algebra of Classes

- Subsets of a set

$$\mathbf{B} \leftrightarrow 2^S$$
$$+ \leftrightarrow \cup$$
$$\cdot \leftrightarrow \cap$$
$$\underline{0} \leftrightarrow \phi$$
$$\underline{1} \leftrightarrow S$$

- S is a universal set ($S \neq \phi$). Each subset of S is called a *class* of S.
- If $S = \{a,b\}$, then **B** = $\{\phi, \{a\}, \{b\}, \{a, b\}\}$
- **B** (= 2^S) is closed under $\cup$ and $\cap$

Algebra of Classes

- Commutative laws: $\forall S_1, S_2 \in 2^S$
  $S_1 \cup S_2 = S_2 \cup S_1$
  $S_1 \cap S_2 = S_2 \cap S_1$
- Distributive laws: $\forall S_1, S_2, S_3 \in 2^S$
  $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$
  $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$
- Identities: $\forall S_1 \in 2^S$
  $S_1 \cup \phi = S_1$
  $S_1 \cap S = S_1$
- Complements: $\forall S_1 \in 2^S$, $\exists S_1' \in 2^S$, $S_1' = S \backslash S_1$ s.t.
  $S_1 \cup S_1' = S$
  $S_1 \cap S_1' = \phi$

Algebra of Classes

- *Stone Representation Theorem*:
  Every finite Boolean algebra is isomorphic to the Boolean algebra of subsets of some finite set S

  Therefore, for all finite Boolean algebra, **|B|** can only be 2^k for some $k \geq 1$.
- The theorem proves that finite class algebras are not specialized (i.e. no exclusive properties, e.g. $x + y = 1$ *iff* $x=1$ or $y=1$ in two-element Boolean algebra)
  - Can reason in terms of specific and easily "visualizable" concepts (union, intersection, empty set, universal set) rather than abstract operations ($+$, $\cdot$, $\underline{0}$, $\underline{1}$)

Instance 3: Arithmetic Boolean Algebra

- (D_n, *lcm*, *gcd*, 1, n)
  n: product of distinct prime numbers
  D_n: set of all divisors of n
  lcm: least common multiple
  gcd: greatest common divisor
  1: integer 1 (not the Boolean 1-element)
- $n = 30 = 2 \times 3 \times 5$
- $D_n = \{1, 2, 3, 5, 6, 10, 15, 30\}$
- If we look at D_n as $\{\phi, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 3, 5\}\}$, it is easy to see that arithmetic Boolean algebra is isomorphic to the algebra of classes.
  - See Stone Representation Theorem

Instance 4: Algebra of Propositional Functions

- (P, $\vee$, $\wedge$, □, ■)
  P: the set of propositional functions of n given variables
  $\vee$: disjunction symbol (OR)
  $\wedge$: conjunction symbol (AND)
  □: formula that is always false (contradiction)
  ■: formula that is always true (tautology)

Lessons from Abstraction

- Abstract mathematical objects in terms of simple rules
- A systematic way of characterizing various seemingly unrelated mathematical objects
- Abstraction trims off immaterial details and simplifies problem formulation

Properties of Boolean Algebras

- For arbitrary elements a, b, and c in Boolean algebra

1. Associativity
   $a + (b + c) = (a + b) + c$
   $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. Idempotence
   $a + a = a$
   $a \cdot a = a$
3.
   $a + \underline{1} = \underline{1}$
   $a \cdot \underline{0} = \underline{0}$
4. Absorption
   $a + (a \cdot b) = a$
   $a \cdot (a + b) = a$
5. Involution
   $(a')' = a$
6. De Morgan's Laws
   $(a + b)' = a' \cdot b'$
   $(a \cdot b)' = a' + b'$
7.
   $a + a' \cdot b = a + b$
   $a \cdot (a' + b) = a \cdot b$
8. Consensus
   $a \cdot b + a' \cdot c + b \cdot c =$
   $a \cdot b + a' \cdot c$
   $(a + b) \cdot (a' + c) \cdot (b + c) =$
   $(a + b) \cdot (a' + c)$

Principle of Duality

- Every identity on Boolean algebra is transformed into another identity if the following is interchanged
  - the operations + and ·,
  - the elements $\underline{0}$ and $\underline{1}$
- Example:
  - a + $\underline{1}$ = $\underline{1}$
  - a · $\underline{0}$ = $\underline{0}$

Postulates for Boolean Algebra (Revisited)

Duality in (**B**, +, ·, $\underline{0}$, $\underline{1}$)

1. **B** is closed under + and ·
   $\forall a,b \in$ **B**, $a + b \in$ **B** and $a \cdot b \in$ **B**
2. Commutative Laws**:** $\forall a,b \in$ **B**
   $a + b = b + a$
   $a \cdot b = b \cdot a$
3. Distributive laws: $\forall a,b \in$ **B**
   $a + (b \cdot c) = (a + b) \cdot (a + c)$
   $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Identities: $\forall a \in$ **B**
   $\underline{0} + a = a$
   $\underline{1} \cdot a = a$
5. Complements: $\forall a \in$ **B**, $\exists\, a' \in$ **B** s.t.
   $a + a' = \underline{1}$
   $a \cdot a' = \underline{0}$

Two Propositions

1. Let a and b be members of a Boolean algebra. Then
   a = $\underline{0}$ and b = $\underline{0}$ iff a + b = $\underline{0}$
   a = $\underline{1}$ and b = $\underline{1}$ iff ab = $\underline{1}$

   c.f. The following two propositions are only true for two-element Boolean algebra (not other Boolean algebra)
   x+y =1 iff x=1 or y=1
   xy=0 iff x=0 or y=0

   Why?

2. Let a and b be members of a Boolean algebra. Then
   a = b iff a′b + ab′ = $\underline{0}$

Boolean Formulas and Boolean Functions

Boolean Formulas and Boolean Functions

- Outline:
  - Definition of Boolean formulas
  - Definition of Boolean functions
  - Boole's expansion theorem
  - The minterm canonical form

n-variable Boolean Formulas

- Given a Boolean algebra **B** and n symbols $x_1, \ldots, x_n$, the set of all Boolean formulas on the n symbols is defined by:
  1. The elements of **B** are Boolean formulas.
  2. The variable symbols $x_1, \ldots, x_n$ are Boolean formulas.
  3. If g and h are Boolean formulas, then so are
     - $(g) + (h)$
     - $(g) \cdot (h)$
     - $(g)'$
  4. A string is a Boolean formula if and only if it is obtained by finitely many applications of rules 1, 2, and 3.
- There are infinitely many n-variable Boolean formulas.

n-variable Boolean Functions

- A Boolean function is a mapping that can be described by a Boolean formula.
- Given an n-variable Boolean formula F, the corresponding n-variable function $f : \mathbf{B}^n \to \mathbf{B}$ is defined as follows:
  1. If F = b $\in$ **B**, then the formula represents the constant function defined by
     $f(x_1,\ldots,x_n) = b \quad \forall([x_1],\ldots,[x_n]) \in \mathbf{B}^n$
  2. If F = x_i, then the formula represents the projection function defined by
     $f(x_1,\ldots,x_n) = x_i \quad \forall([x_1],\ldots,[x_n]) \in \mathbf{B}^n$

     where $[x_k]$ denotes a valuation of variable x_k

n-variable Boolean Functions

3. If the formula is of type either G + H, GH, or G′, then the corresponding n-variable function is defined as follows
   $(g + h)(x_1,\ldots,x_n) = g(x_1,\ldots,x_n) + h(x_1,\ldots,x_n)$
   $(g \cdot h)(x_1,\ldots,x_n) = g(x_1,\ldots,x_n) \cdot h(x_1,\ldots,x_n)$
   $(g')(x_1,\ldots,x_n) = g(x_1,\ldots,x_n)'$

   for $\forall\ ([x_1],\ldots,[x_n]) \in \mathbf{B}^n$

- The number of n-variable Boolean functions over a finite Boolean algebra **B** is *finite.*

Example

- $\mathbf{B} = \{\underline{0}, \underline{1}, a, a'\}$
- Variable symbols: $\{x, y\}$
- 2-variable Boolean formula: e.g., $a' x + a y'$
- 2-variable Boolean function: $f : \mathbf{B}^2 \rightarrow \mathbf{B}$
- Domain $\mathbf{B}^2 = \{(\underline{0},\underline{0}), (\underline{0},\underline{1}), \ldots, (a,a)\}$

x	*y*	*f*
$\underline{0}$	$\underline{0}$	a
$\underline{0}$	$\underline{1}$	$\underline{0}$
$\underline{0}$	a′	a
$\underline{0}$	a	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$
$\underline{1}$	$\underline{1}$	a'
$\underline{1}$	a′	$\underline{1}$
$\underline{1}$	a	a'
a	$\underline{0}$	a
a	$\underline{1}$	$\underline{0}$
a	a′	a
a	a	$\underline{0}$
a′	$\underline{0}$	$\underline{1}$
a′	$\underline{1}$	a'
a′	a′	$\underline{1}$
a′	a	a'

Boole's Expansion Theorem

Theorem 1 If $f : \mathbf{B}^n \rightarrow \mathbf{B}$ is a Boolean function, then

$$f(x_1,\ldots,x_n) = x'_1\, f(\underline{0},\ldots,x_n) + x_1\, f(\underline{1},\ldots,x_n)$$

for $\forall\ ([x_1],\ldots,[x_n]) \in \mathbf{B}^n$

Proof. Case analysis of Boolean functions under the construction rules. Apply postulates of Boolean algebra.

- The theorem holds not only for two-element Boolean algebra (c.f. Shannon expansion)

Minterm Canonical Form

Theorem 2 A function $f : \mathbf{B}^n \rightarrow \mathbf{B}$ is Boolean if and only if it can be expressed in the minterm canonical form

$$f(X) = \sum_{A \in \{\underline{0},\underline{1}\}^n} f(A) \cdot X^A$$

where $X = (x_1, \ldots, x_n) \in \mathbf{B}^n$, $A = (a_1,\ldots,a_n) \in \{\underline{0},\underline{1}\}^n$, and $X^A \equiv x_1^{a1} \cdot x_2^{a2} \cdots x_n^{an}$ (with $x^{\underline{0}} \equiv x'$ and $x^{\underline{1}} \equiv x$)

Proof.

($\Rightarrow$) Follows from Boole's expansion theorem.

($\Leftarrow$) Examine the construction rules of Boolean functions.

Example

f is not Boolean!

Proof. If f is Boolean, f can be expressed by $f(x) = x\, f(1) + x'\, f(0) = x + a\, x'$ from the minterm canonical form. However, substituting x = a in the previous expression yields: $f(a) = a + a\, a' = a \neq 1$

x	f(x)
0	a
1	1
a′	a′
a	1

Why Study General Boolean Algebra?

- General algebras can't be avoided

  $f = x\, y + x\, z' + x'\, z$

  - Two-element view: $x, y, z \in \{0,1\}$ and $f \in \{0,1\}$
  - General algebra view: *f* as a member of the Boolean algebra of 3-variable Boolean functions

Why Study General Boolean Algebra?

- General algebras are useful
  - Two-element view: Truth tables include only 0 and 1.
  - General algebra view: Truth tables contain any elements of **B**.

J	K	Q	Q+
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
..	..	..	..

J	K	Q+
0	0	Q
0	1	0
1	0	1
1	1	Q′