

Logic Synthesis and Verification

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1

Boolean Function Representation

Reading:
Logic Synthesis in a Nutshell
Section 2

most of the following slides are by
courtesy of Andreas Kuehlmann

2

Assumption

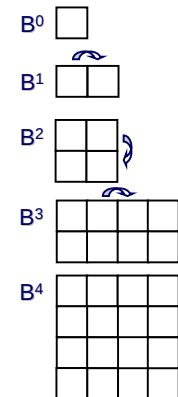
- Unless otherwise said, from now on we are concerned with two-element Boolean algebra (i.e. $\mathbf{B} = \{0,1\}$)

3

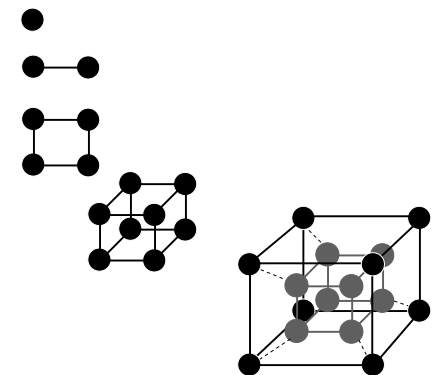
Boolean Space

- $\mathbf{B} = \{0,1\}$
- $\mathbf{B}^2 = \{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$

Karnaugh Maps:



Boolean Lattices:



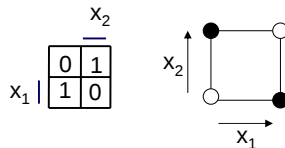
4

Boolean Function

- For $\mathbf{B} = \{0,1\}$, a Boolean function $f: \mathbf{B}^n \rightarrow \mathbf{B}$ over variables $x_1, \dots, x_n$ maps each Boolean valuation (truth assignment) in $\mathbf{B}^n$ to 0 or 1

Example

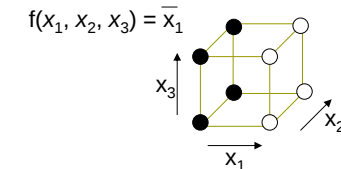
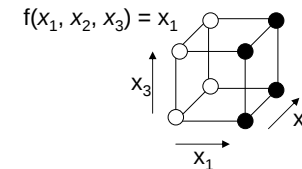
$f(x_1, x_2)$ with $f(0,0) = 0$, $f(0,1) = 1$, $f(1,0) = 1$, $f(1,1) = 0$



5

Boolean Function

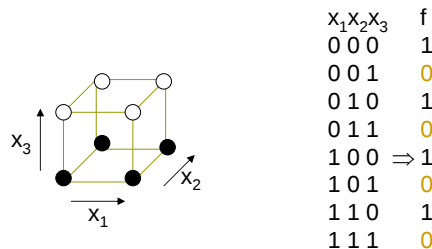
- Onset** of f , denoted as f^1 , is $f^1 = \{v \in \mathbf{B}^n \mid f(v) = 1\}$
 - If $f^1 = \mathbf{B}^n$, f is a **tautology**
- Offset** of f , denoted as f^0 , is $f^0 = \{v \in \mathbf{B}^n \mid f(v) = 0\}$
 - If $f^0 = \mathbf{B}^n$, f is **unsatisfiable**. Otherwise, f is **satisfiable**.
- f^1 and f^0 are sets, not functions!
- Boolean functions f and g are **equivalent** if $\forall v \in \mathbf{B}^n. f(v) = g(v)$ where v is a truth assignment or Boolean valuation
- A **literal** is a Boolean variable x or its negation x' (or $x, \neg x$) in a Boolean formula



6

Boolean Function

- There are 2^n vertices in $\mathbf{B}^n$
- There are 2^{2^n} distinct Boolean functions
 - Each subset $f^1 \subseteq \mathbf{B}^n$ of vertices in $\mathbf{B}^n$ forms a distinct Boolean function f with onset f^1



$x_1 x_2 x_3$	f
000	1
001	0
010	1
011	0
100 $\Rightarrow$	1
101	0
110	1
111	0

7

Boolean Operations

Given two Boolean functions:

$$f: \mathbf{B}^n \rightarrow \mathbf{B}$$

$$g: \mathbf{B}^n \rightarrow \mathbf{B}$$

- $h = f \wedge g$ from **AND** operation is defined as
 $h^1 = f^1 \cap g^1$; $h^0 = \mathbf{B}^n \setminus h^1$
- $h = f \vee g$ from **OR** operation is defined as
 $h^1 = f^1 \cup g^1$; $h^0 = \mathbf{B}^n \setminus h^1$
- $h = \neg f$ from **COMPLEMENT** operation is defined as
 $h^1 = f^0$; $h^0 = f^1$

8

Cofactor and Quantification

Given a Boolean function:

$f: \mathbf{B}^n \rightarrow \mathbf{B}$, with the input variable $(x_1, x_2, \dots, x_i, \dots, x_n)$

- **Positive cofactor on variable x_i**
 $h = f_{x_i}$ is defined as $h = f(x_1, x_2, \dots, 1, \dots, x_n)$
- **Negative cofactor on variable x_i**
 $h = f_{\neg x_i}$ is defined as $h = f(x_1, x_2, \dots, 0, \dots, x_n)$
- **Existential quantification over variable x_i**
 $h = \exists x_i. f$ is defined as $h = f(x_1, x_2, \dots, 0, \dots, x_n) \vee f(x_1, x_2, \dots, 1, \dots, x_n)$
- **Universal quantification over variable x_i**
 $h = \forall x_i. f$ is defined as $h = f(x_1, x_2, \dots, 0, \dots, x_n) \wedge f(x_1, x_2, \dots, 1, \dots, x_n)$
- **Boolean difference over variable x_i**
 $h = \partial f / \partial x_i$ is defined as $h = f(x_1, x_2, \dots, 0, \dots, x_n) \oplus f(x_1, x_2, \dots, 1, \dots, x_n)$

9

Representation of Boolean Function

- Represent Boolean functions for two reasons
 - to represent and manipulate the **actual circuit we are implementing**
 - to facilitate **Boolean reasoning**
- Data structures for representation
 - **Truth table**
 - **Boolean formula**
 - Sum of products (Disjunctive “normal” form, DNF)
 - Product of sums (Conjunctive “normal” form, CNF)
 - **Boolean network**
 - Circuit (network of Boolean primitives)
 - And-inverter graph (AIG)
 - **Binary Decision Diagram (BDD)**

10

Boolean Function Representation Truth Table

- Truth table (function table for multi-valued functions):

The **truth table** of a function $f: \mathbf{B}^n \rightarrow \mathbf{B}$ is a tabulation of its value at each of the 2^n vertices of $\mathbf{B}^n$.

In other words the truth table lists all **minterms**

Example: $f = a'b'c'd + a'b'cd + a'bc'd + ab'c'd + ab'cd + abc'd + abcd' + abcd$

The truth table representation is

- impractical for large n
- canonical

If two functions are the same, then their **canonical** representations are isomorphic.

	abcd	f		abcd	f
0	0000	0	8	1000	0
1	0001	1	9	1001	1
2	0010	0	10	1010	0
3	0011	1	11	1011	1
4	0100	0	12	1100	0
5	0101	1	13	1101	1
6	0110	0	14	1110	1
7	0111	0	15	1111	1

11

Boolean Function Representation Boolean Formula

- A **Boolean formula** is defined inductively as an expression with the following formation rules (syntax):

formula ::=	'(' formula ')'	
	Boolean constant	(true or false)
	Boolean variable	
	formula “+” formula	(OR operator)
	formula “.” formula	(AND operator)
	¬ formula	(complement)

Example

$$f = (x_1 \cdot x_2) + (x_3) + \neg(\neg(x_4 \cdot (\neg x_1)))$$

typically “.” is omitted and ‘(, ’ and ‘¬’ are simply reduced by priority,

e.g. $f = x_1 x_2 + x_3 + x_4 \neg x_1$

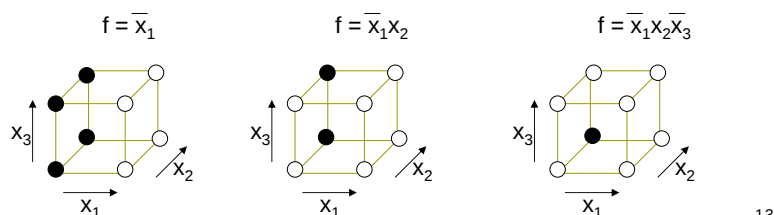
12

Boolean Function Representation Boolean Formula in SOP

- A **cube** is defined as a **conjunction of literals**, i.e. a **product term**.

Example

$C = x_1x_2'x_3$ represents the function with onset: $f^1 = \{(x_1=1, x_2=0, x_3=1)\}$ in the Boolean space spanned by x_1, x_2, x_3 , or $f^1 = \{(x_1=1, x_2=0, x_3=1, x_4=0), (x_1=1, x_2=0, x_3=1, x_4=1)\}$ in the Boolean space spanned by x_1, x_2, x_3, x_4 , or ...



13

Boolean Function Representation Boolean Formula in SOP

- If $C \subseteq f^1$, C the onset of a cube c , then c is an **implicant** of f
- If $C \subseteq \mathbf{B}^n$, and c has k literals, then $|C| = 2^{n-k}$, i.e., C has 2^{n-k} elements

Example

$$c = xy' \quad (c: \mathbf{B}^3 \rightarrow \mathbf{B}), \quad C = \{100, 101\} \subseteq \mathbf{B}^3$$

$$k = 2, \quad n = 3 \quad |C| = 2 = 2^{3-2}$$

- An **implicant** with n literals is a **minterm**

14

Boolean Function Representation Boolean Formula in SOP

- A function can be represented by a **sum-of-cubes** (products):

$$f = ab + ac + bc$$

Since each cube is a product of literals, this is a **sum-of-products (SOP)** representation or **disjunctive normal form (DNF)**
- An SOP can be thought of as a set of cubes F

$$F = \{ab, ac, bc\}$$
- A set of cubes that represents f is called a **cover** of f .

$$F_1 = \{ab, ac, bc\} \quad \text{and} \quad F_2 = \{abc, abc', ab'c, a'bc\}$$

are covers of

$$f = ab + ac + bc.$$
- Mainly used in circuit synthesis; seldom used in Boolean reasoning

15

Boolean Function Representation Boolean Formula in POS

- **Product-of-sums (POS)**, or **conjunctive normal form (CNF)**, representation of Boolean functions
 - Dual of the SOP representation
- Example

$$\varphi = (a+b'+c)(a'+b+c)(a+b'+c')(a+b+c)$$
- A Boolean function in a POS representation can be derived from an SOP representation with De Morgan's law and the distributive law
- Mainly used in Boolean reasoning; rarely used in circuit synthesis (due to the asymmetric characteristics of NMOS and PMOS, and the easiness of adding design constraints)

16

Boolean Function Representation Boolean Network

- Used for two main purposes
 - as target structure for logic implementation which gets restructured in a series of logic synthesis steps until result is acceptable
 - as representation for Boolean reasoning engine
- Efficient representation for most Boolean problems
 - memory complexity is similar to the size of circuits that we are actually building
- Close to the input and output representations of logic synthesis

17

Boolean Function Representation Boolean Network

A **Boolean network** is a directed graph $C(G,N)$ where G are the gates and $N \subseteq (G \times G)$ are the directed edges (nets) connecting the gates.

Some of the vertices are designated:

Inputs: $I \subseteq G$

Outputs: $O \subseteq G$

$I \cap O = \emptyset$

Each gate g is assigned a Boolean function f_g which computes the output of the gate in terms of its inputs.

18

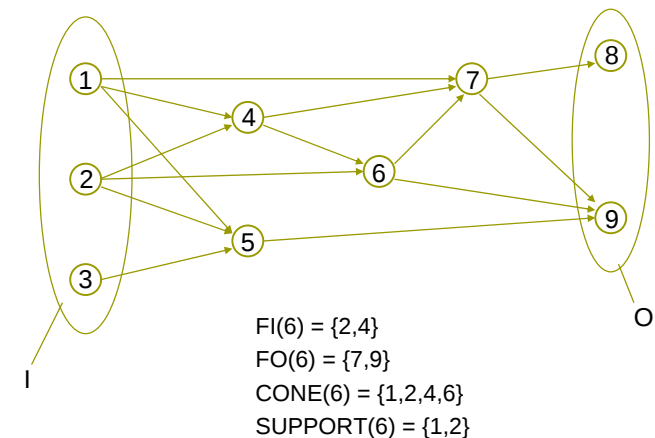
Boolean Function Representation Boolean Network

- The **fanin** $FI(g)$ of a gate g are the predecessor gates of g :
 $FI(g) = \{g' \mid (g',g) \in N\}$ (N : the set of nets)
- The **fanout** $FO(g)$ of a gate g are the successor gates of g :
 $FO(g) = \{g' \mid (g,g') \in N\}$
- The **cone** $CONE(g)$ of a gate g is the **transitive fanin (TFI)** of g and g itself
- The **support** $SUPPORT(g)$ of a gate g are all inputs in its cone:
 $SUPPORT(g) = CONE(g) \cap I$

19

Boolean Function Representation Boolean Network

Example



20

Boolean Function Representation Boolean Network

- Circuit functions are defined recursively:

$$h_{g_i} = \begin{cases} x_i & \text{if } g_i \in I \\ f_{g_i}(h_{g_j}, \dots, h_{g_k}), & g_j, \dots, g_k \in FI(g_i) \text{ otherwise} \end{cases}$$

If G is implemented using physical gates with positive (bounded) delays for their evaluation, the computation of h_g depends in general on those delays.

Definition

A circuit C is called **combinational** if for each input assignment of C for $t \rightarrow \infty$ the evaluation of h_g for all outputs is independent of the internal state of C.

Proposition

A circuit C is combinational if it is acyclic. (converse not true!)

21

Boolean Function Representation Boolean Network

General Boolean network:

- Vertex can have an arbitrary finite number of inputs and outputs
- Vertex can represent any Boolean function stored in different ways, such as:
 - SOPs (e.g. in SIS, a logic synthesis package)
 - BDDs (to be introduced)
 - AIGs (to be introduced)
 - truth tables
 - Boolean expressions read from a library description
 - other sub-circuits (hierarchical representation)
- The data structure allows general manipulations for insertion and deletion of vertices, pins (connection ports of vertices), and nets
 - general but far too slow for Boolean reasoning

22

Boolean Function Representation Boolean Network

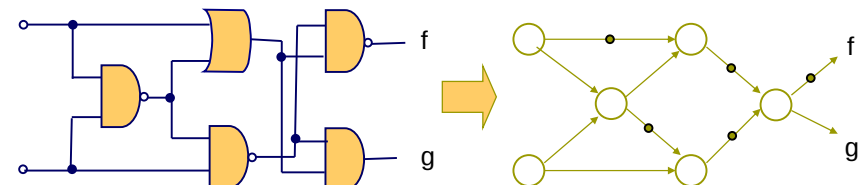
Specialized Boolean network:

- Non-canonical representation in general
 - computational effort of Boolean reasoning is due to this non-canonicity (c.f. BDDs)
- Vertices have fixed number of inputs (e.g. two)
- Vertex function is stored as label (e.g. OR, AND, XOR)
- Allow on-the-fly compaction of circuit structure
 - Support incremental, subsequent reasoning on multiple problems

23

Boolean Function Representation And-Inverter Graph

- AND-INVERTER graphs (AIGs)
 - vertices: 2-input AND gates
 - edges: interconnects with (optional) dots representing INVs
- Hash table to identify and reuse structurally isomorphic circuits



24

Boolean Function Representation And-Inverter Graph

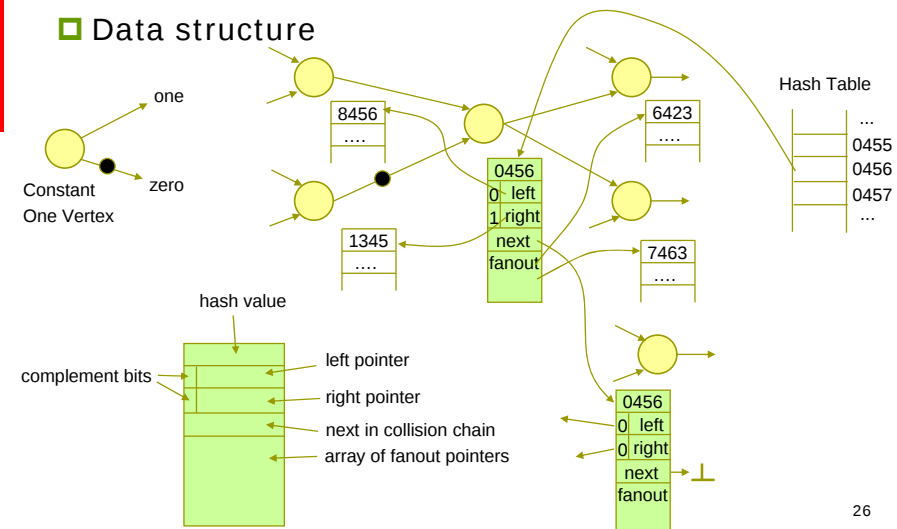
□ Data structure for implementation

- Vertex:
 - pointers (integer indices) to left- and right-child and fanout vertices
 - collision chain pointer
 - other data
- Edge:
 - pointer or index into array
 - one bit to represent inversion
- Global hash table holds each vertex to identify isomorphic structures
- Garbage collection to regularly free un-referenced vertices

25

Boolean Function Representation And-Inverter Graph

□ Data structure



26

Boolean Function Representation And-Inverter Graph

□ AIG package for Boolean reasoning

Engine application:

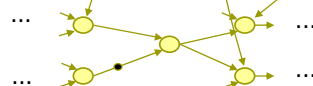
- traverse problem data structure and build Boolean problem using the interface
- call SAT to make decision

Engine Interface:

```
void INIT()
void QUIT()
Edge VAR()
Edge AND(Edge p1,
         Edge p2)
Edge NOT(Edge p1)
Edge OR(Edge p1,
       Edge p2)
...
int SAT(Edge p1)
```

External reference pointers attached to application data structures

Engine Implementation:



27

Boolean Function Representation And-Inverter Graph

□ Hash table look-up

```
Algorithm HASH_LOOKUP(Edge p1, Edge p2) {
    index = HASH_FUNCTION(p1,p2)
    p = &hash_table[index]
    while(p != NULL) {
        if(p->left == p1 && p->right == p2) return p;
        p = p->next;
    }
    return NULL;
}
```

□ Tricks:

- keep collision chain sorted by the address (or index) of p
- use memory locations (or array indices) in topological order for better cache performance

28

Boolean Function Representation And-Inverter Graph

AND operation

```
Algorithm AND(Edge p1, Edge p2){
  if(p1 == const1) return p2
  if(p2 == const1) return p1
  if(p1 == p2)      return p1
  if(p1 == ¬p2)     return const0
  if(p1 == const0 || p2 == const0) return const0

  if(RANK(p1) > RANK(p2)) SWAP(p1, p2)

  if((p = HASH_LOOKUP(p1, p2)) return p
  return CREATE_AND_VERTEX(p1, p2)
}
```

29

Boolean Function Representation And-Inverter Graph

NOT operation

```
Algorithm NOT(Edge p) {
  return TOGGLE_COMPLEMENT_BIT(p)
}
```

OR operation

```
Algorithm OR(Edge p1, Edge p2){
  return (NOT(AND(NOT(p1), NOT(p2))))
}
```

30

Boolean Function Representation And-Inverter Graph

Cofactor operation

```
Algorithm POSITIVE_COFACTOR(Edge p, Edge v){
  if(IS_VAR(p)) {
    if(p == v) {
      if(IS_INVERTED(v) == IS_INVERTED(p)) return const1
      else return const0
    }
    else return p
  }
  if((c = GET_COFACTOR(p, v)) == NULL) {
    left = POSITIVE_COFACTOR(p->left, v)
    right = POSITIVE_COFACTOR(p->right, v)
    c = AND(left, right)
    SET_COFACTOR(p, v, c)
  }
  if(IS_INVERTED(p)) return NOT(c)
  else return c
}
```

31

Boolean Function Representation And-Inverter Graph

Similar algorithm for **NEGATIVE_COFACTOR**

Existential and universal quantifications can be built from AND, OR and COFACTORS

Exercise: Prove $(f \cdot g)_v = f_v \cdot g_v$ and $(\neg f)_v = \neg(f_v)$

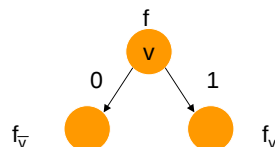
Question: What is the worst-case complexity of performing quantifications over AIGs?

32

Boolean Function Representation Binary Decision Diagram (BDD)

□ A graphical representation of Boolean function

- BDD is a Shannon cofactor tree:
 - $f = v f_v + v' f_{v'}$ (Shannon expansion)
 - vertices represent decision nodes (i.e. multiplexers) controlled by variables
 - leaves are constants "0" and "1"
 - two children of a vertex of f represent two subfunctions f_v and $f_{v'}$
- Variable ordering restriction and reduction rules make (ROBDD) representation **canonical**



33

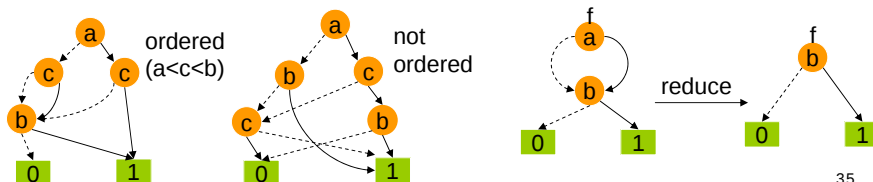
Boolean Function Representation BDD – Canonicalization

- General idea:
 - instead of exploring sub-cases by enumerating them in time, try to store sub-cases in memory
 - **KEY:** two hashing mechanisms:
 - unique table: find identical sub-cases and avoid replication
 - computed table: reduce redundant computation of sub-cases
- Represent logic functions as graphs (DAGs):
 - many logic functions can be represented compactly - usually better than SOPs
- Can be made **canonical** (ROBDD)
 - Shift the effort in a Boolean reasoning engine from SAT algorithm to data representation
- Many logic operations can be performed efficiently on BDD's:
 - usually linear in size of input BDDs
 - tautology checking and complement operation are constant time
- BDD size critically depends on variable ordering

34

Boolean Function Representation BDD – Canonicalization

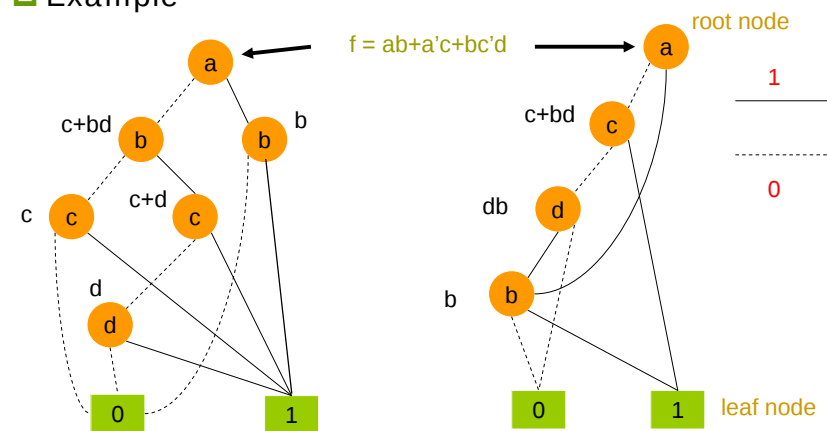
- Directed acyclic graph (DAG)
 - one root node, two terminal-nodes, 0 and 1
 - each node has two children and is controlled by a variable
 - Shannon cofactor tree, except **reduced** and **ordered** (ROBDD)
 - **Ordered:**
 - cofactor variables (splitting variables) in the **same order along all paths**
 - **Reduced:**
 - any node with two identical children is removed
 - two nodes with isomorphic BDD's are merged
- $x_{i_1} < x_{i_2} < x_{i_3} < \dots < x_{i_n}$
- These two rules make any node in an ROBDD represent a distinct logic function



35

Boolean Function Representation BDD

□ Example



Same function with two different variable orders

36

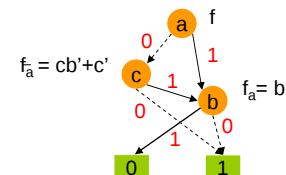
Boolean Function Representation BDD – Canonicity of ROBDD

- Three components make ROBDD canonical (Bryant 1986):
 - unique nodes for constant “0” and “1”
 - identical order of case-splitting variables along each paths
 - a hash table that ensures
 - $(\text{node}(f_v) = \text{node}(g_v)) \wedge (\text{node}(f_{v'}) = \text{node}(g_{v'})) \Rightarrow \text{node}(f) = \text{node}(g)$
 and provides recursive argument that $\text{node}(f)$ is unique when using the unique hash-table

37

Boolean Function Representation BDD – Onset Counting

$F = b' + a'c' = ab' + a'cb' + a'c'$ (all paths to the 1 node)

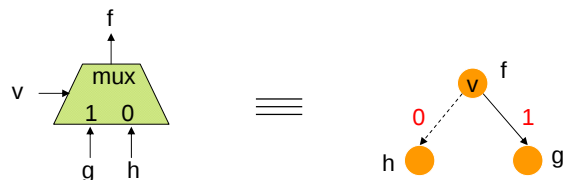


- By tracing all paths to the 1 node, we get a **cover** of **pairwise disjoint** cubes
- BDD does **not** explicitly enumerate all paths; rather it represents paths by a graph whose size is measures by its nodes
 - A DAG can represent an **exponential number of paths** with a linear number of nodes
- BDDs can be used to efficiently represent sets
 - interpret elements of the onset as elements of the set
 - f is called the **characteristic function** of that set

38

Boolean Function Representation BDD – ITE Operator

- Each BDD node can be written as a triplet: $f = \text{ite}(v, g, h)$, where $g = f_v$ and $h = f_{\bar{v}}$, meaning **if v then g else h**



(v is top variable of f)

39

Boolean Function Representation BDD – ITE Operator

- $\text{ite}(f, g, h) = fg + f'h$
 - ITE operator can implement any two variable logic function. There are 16 such functions corresponding to all subsets of vertices of $\mathbf{B}^2$:

Table	Subset	Expression	Equivalent Form
0000	0	0	0
0001	AND(f, g)	$f g$	$\text{ite}(f, g, 0)$
0010	$f > g$	$f g'$	$\text{ite}(f, g', 0)$
0011	f	f	f
0100	$f < g$	$f'g$	$\text{ite}(f, 0, g)$
0101	g	g	g
0110	XOR(f, g)	$f \oplus g$	$\text{ite}(f, g', g)$
0111	OR(f, g)	$f + g$	$\text{ite}(f, 1, g)$
1000	NOR(f, g)	$(f + g)'$	$\text{ite}(f, 0, g')$
1001	XNOR(f, g)	$f \oplus g'$	$\text{ite}(f, g, g')$
1010	NOT(g)	g'	$\text{ite}(g, 0, 1)$
1011	$f \geq g$	$f + g'$	$\text{ite}(f, 1, g')$
1100	NOT(f)	f'	$\text{ite}(f, 0, 1)$
1101	$f \leq g$	$f' + g$	$\text{ite}(f, g, 1)$
1110	NAND(f, g)	$(f g)'$	$\text{ite}(f, g', 1)$
1111	1	1	1

40

Boolean Function Representation BDD – ITE Operator

Recursive operation of ITE

$$\begin{aligned}
 \text{ite}(f, g, h) &= f g + f' h \\
 &= v (f g + f' h)_v + v' (f g + f' h)_{v'} \\
 &= v (f_v g_v + f'_v h_v) + v' (f_{v'} g_{v'} + f'_{v'} h_{v'}) \\
 &= \text{ite}(v, \text{ite}(f_v, g_v, h_v), \text{ite}(f_{v'}, g_{v'}, h_{v'}))
 \end{aligned}$$

■ Let v be the top-most variable of BDDs f, g, h

41

Boolean Function Representation BDD – ITE Operator

Recursive computation of ITE

```

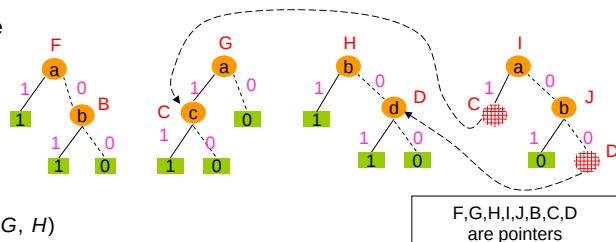
Algorithm ITE(f, g, h)
  if(f == 1) return g
  if(f == 0) return h
  if(g == h) return g

  if((p = HASH_LOOKUP_COMPUTED_TABLE(f,g,h)) return p
  v = TOP_VARIABLE(f, g, h) // top variable from f,g,h
  fn = ITE(f_v, g_v, h_v) // recursive calls
  gn = ITE(f_v, g_v, h_v)
  if(fn == gn) return gn // reduction
  if(!(p = HASH_LOOKUP_UNIQUE_TABLE(v,fn,gn)) {
    p = CREATE_NODE(v,fn,gn) // and insert into UNIQUE_TABLE
  }
  INSERT_COMPUTED_TABLE(p, HASH_KEY{f,g,h})
  return p
}
    
```

42

Boolean Function Representation BDD – ITE Operator

Example



$$\begin{aligned}
 I &= \text{ite}(F, G, H) \\
 &= \text{ite}(a, \text{ite}(F_a, G_a, H_a), \text{ite}(F_{\bar{a}}, G_{\bar{a}}, H_{\bar{a}})) \\
 &= \text{ite}(a, \text{ite}(1, C, H), \text{ite}(B, 0, H)) \\
 &= \text{ite}(a, C, \text{ite}(b, \text{ite}(B_b, 0_b, H_b), \text{ite}(B_{\bar{b}}, 0_{\bar{b}}, H_{\bar{b}}))) \\
 &= \text{ite}(a, C, \text{ite}(b, \text{ite}(1, 0, 1), \text{ite}(0, 0, D))) \\
 &= \text{ite}(a, C, \text{ite}(b, 0, D)) \\
 &= \text{ite}(a, C, J)
 \end{aligned}$$

Check: $F = a + b$
 $G = ac$
 $H = b + d$
 $\text{ite}(F, G, H) = (a + b)(ac) + a'b'(b + d) = ac + a'b'd$

43

Boolean Function Representation BDD – ITE Operator

Tautology checking using ITE

```

Algorithm ITE_CONSTANT(f,g,h) { // returns 0,1, or NC
  if(TRIVIAL_CASE(f,g,h) return result (0,1, or NC)
  if((res = HASH_LOOKUP_COMPUTED_TABLE(f,g,h))) return res
  v = TOP_VARIABLE(f, g, h)
  i = ITE_CONSTANT(f_v, g_v, h_v)
  if(i == NC) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h}) // special table!!
    return NC
  }
  e = ITE_CONSTANT(f_v, g_v, h_v)
  if(e == NC) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h})
    return NC
  }
  if(e != i) {
    INSERT_COMPUTED_TABLE(NC, HASH_KEY{f,g,h})
    return NC
  }
  INSERT_COMPUTED_TABLE(e, HASH_KEY{f,g,h})
  return i;
}
    
```

44

Boolean Function Representation BDD – ITE Operator

Composition using ITE

- **Compose** is an important operation, e.g. for building the BDD of a circuit backwards, $\text{Compose}(F, v, G) : F(v, x) \rightarrow F(G(x), x)$, means substitute $v = G(x)$

```

Algorithm COMPOSE(F,v,G) {
  if(TOP_VARIABLE(F) > v) return F // F does not depend on v
  if(TOP_VARIABLE(F) == v) return ITE(G,F1,F0)
  i = COMPOSE(F1,v,G)
  e = COMPOSE(F0,v,G)
  return ITE(TOP_VARIABLE(F),i,e)
}
    
```

Note:

1. F1 and F0 are the 1-child and 0-child of F, respectively
2. G, i, e are not functions of v
3. If TOP_VARIABLE of F is v, then $\text{ITE}(G, F1, F0)$ does the replacement of v by G

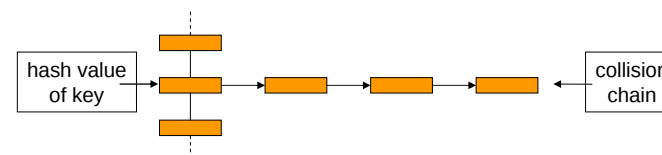
45

Boolean Function Representation BDD – Implementation Issues

Unique table:

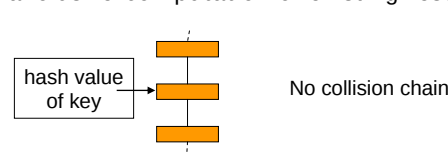
avoids duplication of existing nodes

- Hash-Table: hash-function(key) = value
- identical to the use of a hash-table in AND/INVERTER circuits



Computed table:

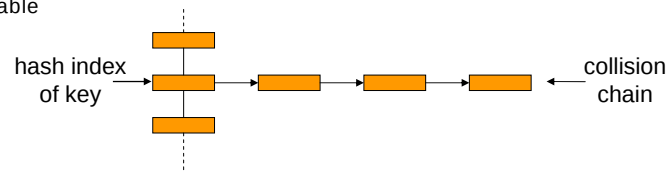
avoids re-computation of existing results



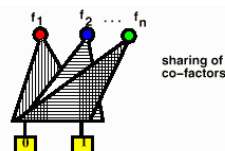
46

Boolean Function Representation BDD – Implementation Issues

Unique table



- Before a node $\text{ite}(v, g, h)$ is added to BDD database, it is looked up in the "unique-table". If it is there, then existing pointer to node is used to represent the logic function. Otherwise, a new node is added to the unique-table and the new pointer returned.
- Thus a **strong canonical form** is maintained. The node for $f = \text{ite}(v, g, h)$ exists iff $\text{ite}(v, g, h)$ is in the unique-table. There is only one pointer for $\text{ite}(v, g, h)$ and that is the address to the unique-table entry.
- Unique-table allows single multi-rooted DAG to represent all users' functions

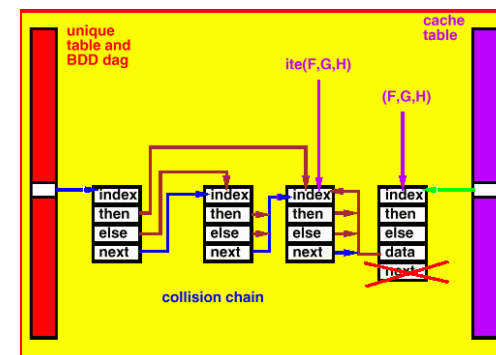


47

Boolean Function Representation BDD – Implementation Issues

Computed table

- Keep a record of (F, G, H) triplets already computed by the ITE operator
 - software cache ("cache" table)
 - simply hash-table without collision chain (lossy cache)



48

Boolean Function Representation BDD – Implementation Issues

□ Use of computed table

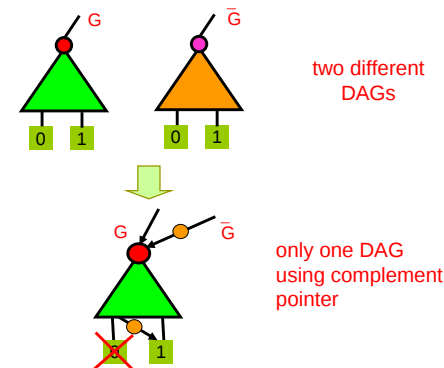
- BDD packages often use optimized implementations for special operations
 - e.g. ITE_Constant (check whether the result would be a constant) AND_Exist (AND operation with existential quantification)
- All operations need a cache for decent performance
 - local cache
 - for one operation only - cache will be thrown away after operation is finished (e.g. AND_Exist)
 - special cache for each operation
 - does not need to store operation type
 - shared cache for all operations
 - better memory handling
 - needs to store operation type

49

Boolean Function Representation BDD – Implementation Issues

□ Complemented edges

- Combine inverted functions by using complemented edges
 - similar to AIG
 - reduces memory requirements
 - more importantly, makes operations NOT, ITE more efficient

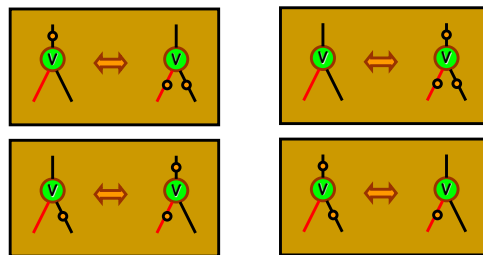


50

Boolean Function Representation BDD – Implementation Issues

□ Complemented edges

- To maintain strong canonical form, need to resolve 4 equivalences:



- **Solution:** Always choose the ones on **left**, i.e. the “then” leg must have **no** complement edge.

51

Boolean Function Representation BDD – Implementation Issues

□ Complemented edges

Standard triples:

$$\begin{aligned} \text{ite}(F, F, G) &\Rightarrow \text{ite}(F, 1, G) \\ \text{ite}(F, G, F) &\Rightarrow \text{ite}(F, G, 0) \\ \text{ite}(F, G, \neg F) &\Rightarrow \text{ite}(F, G, 1) \\ \text{ite}(F, \neg F, G) &\Rightarrow \text{ite}(F, 0, G) \end{aligned}$$

To resolve equivalences:

$$\begin{aligned} \text{ite}(F, 1, G) &\equiv \text{ite}(G, 1, F) \\ \text{ite}(F, 0, G) &\equiv \text{ite}(\neg G, 1, \neg F) \\ \text{ite}(F, G, 0) &\equiv \text{ite}(G, F, 0) \\ \text{ite}(F, G, 1) &\equiv \text{ite}(\neg G, \neg F, 1) \\ \text{ite}(F, G, \neg G) &\equiv \text{ite}(G, F, \neg F) \end{aligned}$$

To maximize matches on computed table:

1. First argument is chosen with smallest top variable.
2. Break ties with smallest address pointer. (breaks PORTABILITY!)

Triples:

$$\text{ite}(F, G, H) \equiv \text{ite}(\neg F, H, G) \equiv \neg \text{ite}(F, \neg G, \neg H) \equiv \neg \text{ite}(\neg F, \neg H, \neg G)$$

Choose the one such that the first and second argument of *ite* should not be complement edges (i.e. the first one above)

52

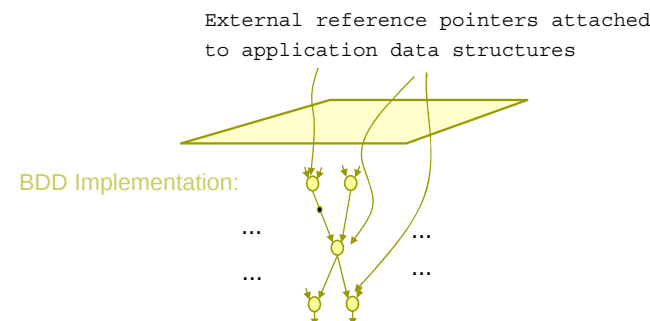
Boolean Function Representation BDD – Implementation Issues

- Variable ordering – static
 - variable ordering is computed up-front based on the problem structure
 - works well for many practical combinational functions
 - general scheme: control variables first
 - DFS order is good for most cases
 - works bad for unstructured problems
 - e.g. using BDDs to represent arbitrary sets
 - lots of ordering algorithms
 - simulated annealing, genetic algorithms
 - give better results but extremely costly

53

Boolean Function Representation BDD – Implementation Issues

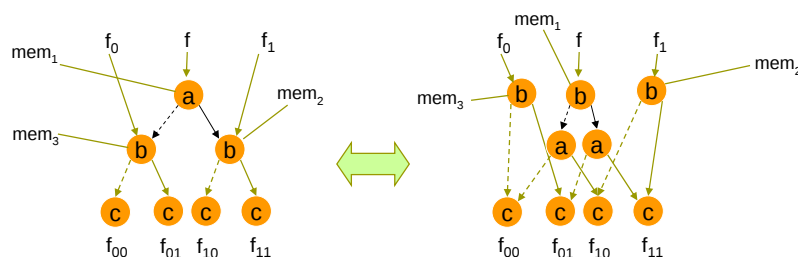
- Variable ordering – dynamic
 - Changes the order in the middle of BDD applications
 - must keep same global order
 - Problem: External pointers reference internal nodes!



54

Boolean Function Representation BDD – Implementation Issues

- Variable ordering – dynamic
 - Theorem (Friedman):**
 - Permuting any top part of the variable order has no effect on the nodes labeled by variables in the bottom part.
 - Permuting any bottom part of the variable order has no effect on the nodes labeled by variables in the top part.
 - Trick: Two adjacent variable layers can be exchanged by keeping the original memory locations for the nodes



55

Boolean Function Representation BDD – Implementation Issues

- Variable ordering – dynamic
 - BDD sifting:
 - shift each BDD variable to the top and then to the bottom and see which position had minimal number of BDD nodes
 - efficient if separate hash-table for each variable
 - can stop if lower bound on size is worse than the best found so far
 - shortcut: two layers can be swapped very cheaply if there is no interaction between them
 - expensive operation
 - grouping of BDD variables:
 - for many applications, grouping variables gives better ordering
 - e.g. current state and next state variables in state traversal
 - grouping variables for sifting

56

Boolean Function Representation BDD – Implementation Issues

□ Garbage collection

- Important to free and reuse memory of unused BDD nodes including
 - those explicitly freed by an external `bdd_free` operation
 - those temporary created during BDD operations
- Two mechanisms to check whether a BDD is not referenced:
 - **Reference counter** at each node
 - increment whenever node gets one more referenced
 - decrement when node gets de-referenced
 - take care of counter-overflow
 - **Mark and sweep** algorithm
 - does not need counter
 - first pass, mark all BDDs that are referenced
 - second pass, free the BDDs that are not marked
 - need additional handle layer for external references

57

Boolean Function Representation BDD – Implementation Issues

□ Garbage collection

- Timing is crucial because garbage collection is expensive
 - immediately when node gets freed
 - bad because dead nodes get often reincarnated in subsequent operations
 - regular garbage collections based on statistics obtained during BDD operations
- Computed-table must be cleared since not used in reference mechanism
- Improving memory locality and therefore cache behavior

58

Boolean Function Representation BDD – Variants

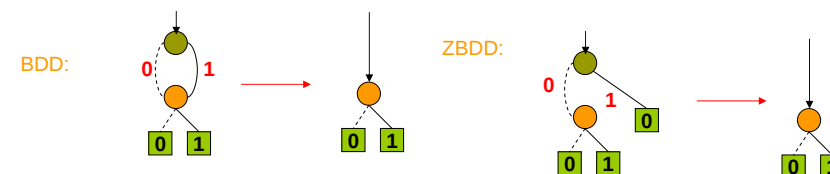
- MDD: Multi-valued DD
 - have more than two branches
 - can be implemented using a regular BDD package with binary encoding
 - the binary variables for one MV variable do not have to stay together and thus potentially better ordering
- ADD: (Algebraic BDDs) MTBDD
 - multi-terminal BDDs
 - decision tree is binary
 - multiple leaves, including real numbers, sets or arbitrary objects
 - efficient for matrix computations and other non-integer applications
- FDD: Free-order BDD
 - variable ordering differs
 - not canonical anymore
- ...

59

Boolean Function Representation BDD – Variants

□ Zero suppressed BDD (ZDD)

- ZBDDs were invented by Minato to efficiently represent **sparse** sets. They have turned out to be **useful** in implicit methods for representing primes (which usually are a sparse subset of all cubes).
- Different reduction rules:
 - **BDD**: eliminate all nodes where **then** edge and **else** edge point to the same node.
 - **ZBDD**: eliminate all nodes where the **then** node points to 0. Connect incoming edges to **else** node.
 - **For both**: share equivalent nodes.

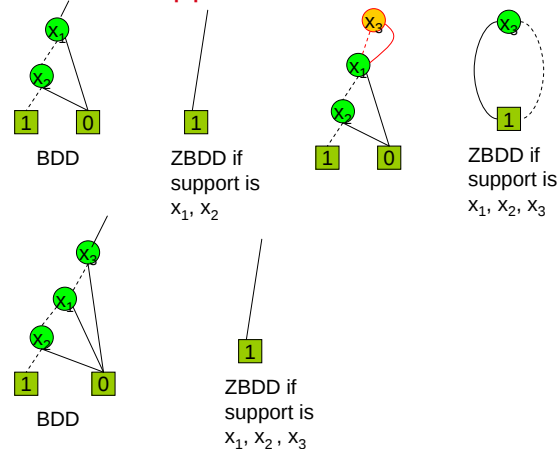


60

Boolean Function Representation BDD – Variants

Theorem: ZBDDs are canonical **given a variable ordering and the support set**

Example



61

Boolean Function Representation Summary

- Sum of products
 - Good for circuit synthesis
- Product of sums
 - Good for Boolean reasoning
- Boolean network
 - Generic network
 - Good for multi-level circuit synthesis
 - And-inverter graph
 - Good for Boolean reasoning
- Binary decision diagram
 - Good for Boolean reasoning

62

Boolean Reasoning

Reading:
Logic Synthesis in a Nutshell
Section 2

most of the following slides are by
courtesy of Andreas Kuehlmann

63

Boolean Reasoning Satisfiability (SAT)

- Boolean reasoning engines need:
 - a data structure to represent problem instances
 - a decision procedure to decide about SAT or UNSAT
- Fundamental tradeoff
 - canonical data structure (e.g. truth table, ROBDD)
 - data structure uniquely represents function
 - decision procedure is trivial (e.g., just pointer comparison)
 - Problem: size of data structure is in general exponential
 - non-canonical data structure (e.g. AIG, CNF)
 - systematic search for satisfying assignment
 - size of data structure is linear
 - Problem: decision may take an exponential amount of time

64

Boolean Reasoning SAT

□ Basic SAT algorithms:

- branch and bound algorithm
 - branching on the assignments of primary inputs only or those of all variables
 - E.g. PODEM vs. D-algorithms in ATPG

□ Basic data structures:

- circuits or CNF formulas
- SAT on circuits is **identical** to the justification part in ATPG
 - 1st half of ATPG: justification
 - find an input assignment that forces an internal signal to a required value
 - 2nd half of ATPG: propagation
 - make a signal change at an internal signal observable at some outputs (can be easily formulated as SAT over CNF formulas)

65

Boolean Reasoning SAT vs. Tautology

□ SAT:

- find a truth assignment to the inputs making a given Boolean formula true
- NP-complete

□ Tautology:

- find a truth assignment to the inputs making a given Boolean formula false
- coNP-complete

□ SAT and Tautology are dual to each other

- checking SAT on formula ϕ = checking Tautology on formula $\neg\phi$, and vice versa

66

Boolean Reasoning SAT – AIG-based Decision Procedure

□ General Davis-Putnam procedure

- search for **consistent** assignment to entire cone of requested vertex in AIG by systematically trying all combinations (**may be partial**)
- keep a queue of vertices that remain to be justified
 - pick **decision vertex** from the queue and case split on possible assignments
 - for each case
 - propagate as many implications as possible
 - generate more vertices to be justified
 - if conflicting assignment encountered, undo all implications and backtrack
 - recur to next vertex from queue

67

Boolean Reasoning SAT – AIG-based Decision Procedure

□ General Davis-Putnam procedure

```
Algorithm SAT(Edge p) {  
    queue = INIT_QUEUE(p)  
    if(!IMPLY(p)) return FALSE  
    return JUSTIFY(queue)  
}
```

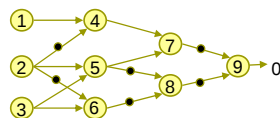
```
Algorithm JUSTIFY(queue) {  
    if(QUEUE_EMPTY(queue)) return TRUE  
    mark = ASSIGNMENT_MARK()  
    v = QUEUE_NEXT(queue) // decision vertex  
    if(IMPLY(NOT(v))) {  
        if(JUSTIFY(queue)) return TRUE  
    } // conflict  
    UNDO_ASSIGNMENTS(mark)  
    if(IMPLY(v)) {  
        if(JUSTIFY(queue)) return TRUE  
    } // conflict  
    UNDO_ASSIGNMENTS(mark)  
    return FALSE  
}
```

68

Boolean Reasoning SAT – AIG-based Decision Procedure

Example

SAT(NOT(9))??



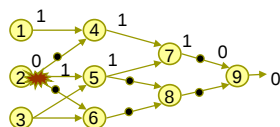
Queue

Assignments

9

9

1st case for 9:



conflict !

- undo all assignments
- backtrack

9

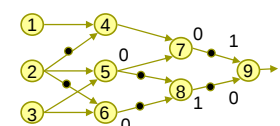
9
7
4
5
1
1
2

69

Boolean Reasoning SAT – AIG-based Decision Procedure

Example (cont'd)

2nd case for 9:



Note:

vertex 7 is justified
by 8->5->7

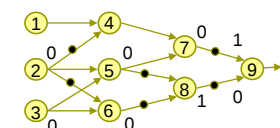
Queue

Assignments

5
6

9
7
8
5
6

1st case for 5:



Solution cube: 1 = x, 2 = 0, 3 = 0

9
7
8
5
6
2
3

70

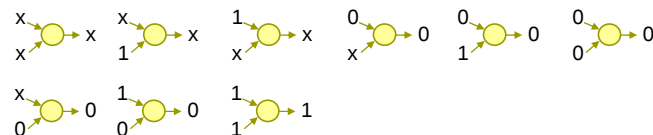
Boolean Reasoning SAT – AIG-based Decision Procedure

Implication

- Fast implication procedure is key for efficient SAT solver!
 - don't move into circuit parts that are not sensitized to current SAT problem
 - detect conflicts as early as possible

Table lookup implementation (27 cases):

No-implication cases:



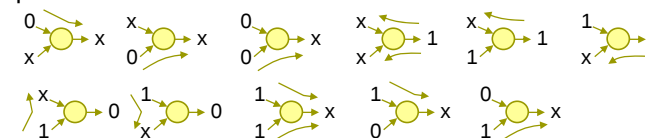
71

Boolean Reasoning SAT – AIG-based Decision Procedure

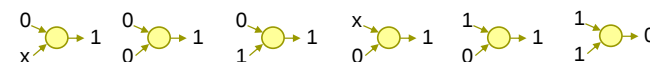
Implication (cont'd)

Table lookup implementation (27 cases):

Implication cases:



Conflict cases:



Split case:

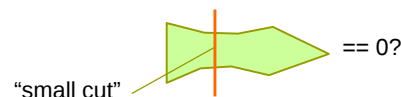


72

Boolean Reasoning SAT – AIG-based Decision Procedure

Case split

- Different heuristics work well for particular problem classes
- Often **depth-first heuristic** is good because it generates conflicts quickly
- Mixture of **depth-first** and **breadth-first** schedule
- Other heuristics:
 - pick the vertex with the **largest fanout**
 - count the **polarities** of the fanout **separately** and pick the vertex with the highest count in either polarity
 - run a **full implication** phase on all outstanding case splits and count the number of implications one would get
 - pick vertices that are involved in **small cut** of the circuit

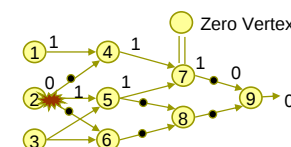


73

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning

- Learning is the process of adding "shortcuts" to the circuit structure that avoids case splits
 - static learning:
 - global implications are learned
 - dynamic learning:
 - learned implications only hold in current part of the search tree
- Learned implications are stores as additional network
- Example (cont'd)
 - 1st case for vertex 9 lead to conflict
 - If we were to try the same assignment again (e.g. for the next SAT call), we would get the same conflict => merge vertex 7 with zero-vertex



- if rehashing is invoked
vertex 9 is simplified and
merged with vertex 8

74

Boolean Reasoning SAT – AIG-based Decision Procedure

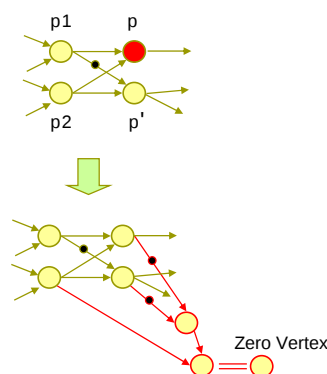
Learning – static

- Implications that can be learned structurally from the circuit
 - Add learned structure as circuit

Use hash table to find structure in circuit:

```

Algorithm CREATE_AND(p1, p2) {
    ... // create new vertex p
    if((p'=HASH_LOOKUP(p1, NOT(p2))) {
        LEARN(((p=0)&(p'=0)) => (p1=0))
    }
    if((p'=HASH_LOOKUP(NOT(p1), p2)) {
        LEARN(((p=0)&(p'=0)) => (p2=0))
    }
}
    
```

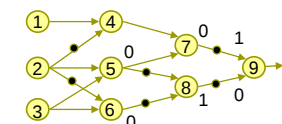


75

Boolean Reasoning SAT – AIG-based Decision Procedure

Example (cont'd)

2nd case for 9 (original):

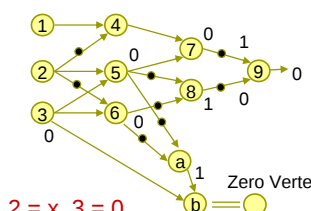


Queue Assignments

5
6

9
7
8
5
6

2nd case for 9 (with static learning):



9
7
8
5
6
a
3

Solution cube: 1 = x, 2 = x, 3 = 0

76

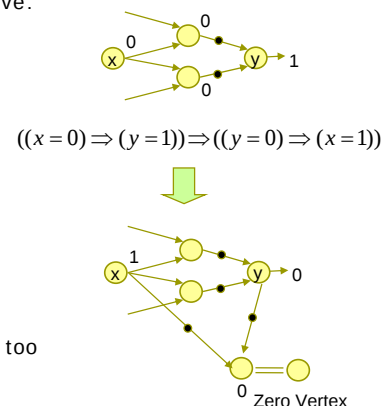
Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – static

- Other learning based on contra-positive:
if $(P \Rightarrow Q)$, then $(\neg Q \Rightarrow \neg P)$

```
foreach vertex v {
  mark = ASSIGNMENT_MARK()
  IMPLY(v)
  LEARN_IMPLICATIONS(v)
  UNDO_ASSIGNMENTS(mark)
  IMPLY(NOT(v))
  LEARN_IMPLICATIONS(NOT(v))
  UNDO_ASSIGNMENTS(mark)
}
```

- Problem: learned implications are far too many
 - solution: restrict learning to non-trivial implications and filter redundant implications



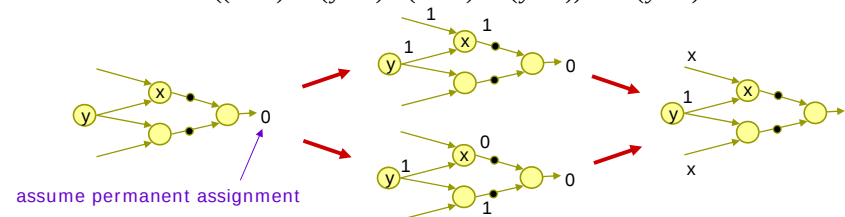
77

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – static and recursive

- Compute the set of all implications for both case splits on level i
 - Static learning of constants, equivalences
- Intersect both split cases to learn for level $i-1$

$$((x=1) \Rightarrow (y=1) \wedge (x=0) \Rightarrow (y=1)) \Rightarrow (y=1)$$



- Apply learning recursively until all case splits exhausted
 - recursive learning is complete but very expensive in practice for levels $> 2, 3$
 - restrict learning level to fixed number $\rightarrow$ becomes incomplete

78

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – static and recursive

```
Algorithm RECURSIVE_LEARN(int level) {
  if(v = PICK_SPLITTING_VERTEX()) {
    mark = ASSIGNMENT_MARK()
    IMPLY(v)
    IMPL1 = RECURSIVE_LEARN(level+1)
    UNDO_ASSIGNMENTS(mark)
    IMPLY(NOT(v))
    IMPL0 = RECURSIVE_LEARN(level+1)
    UNDO_ASSIGNMENTS(mark)
    return IMPL1  $\cap$  IMPL0
  }
  else { // completely justified
    return IMPLICATIONS
  }
}
```

79

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – dynamic

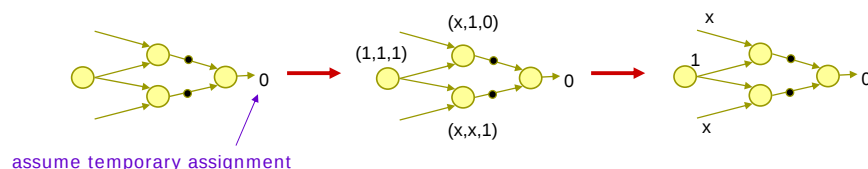
- Learn implications in a sub-tree of searching
 - cannot simply add permanent structure because not globally valid
 - add and remove learned structure (expensive)
 - add branching condition to the learned implication
 - of no use unless we prune the condition (conflict learning)
 - use implication and assignment mechanism to assign and undo assigns
 - e.g., dynamic recursive learning with fixed recursion level
 - Dynamic learning of equivalence relations (Stalmarck procedure)
 - learn equivalence relations by dynamically rewriting the formula

80

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – dynamic

- Efficient implementation of **dynamic recursive learning** with level 1:
 - consider both sub-cases in parallel
 - use 27-valued logic in the IMPLY routine
(level0-value, level1-choice1, level1-choice2)
({0,1,x}, {0,1,x}, {0,1,x})
 - automatically set learned values for level0 if both level1 choices agree, e.g.,



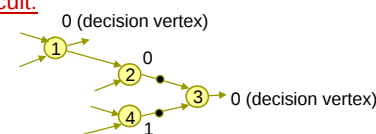
81

Boolean Reasoning SAT – AIG-based Decision Procedure

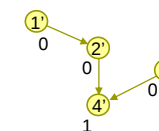
Learning – conflict-based (c.f. structure-based)

- Idea:** Learn the situation under which a particular conflict occurred and assert it to 0
 - IMPLY will use this “shortcut” to detect similar conflict earlier
- Definition:** An **implication graph** is a directed Graph $I(G', E)$, $G' \subseteq G$ are the gates of C with assigned values $v_g \neq \text{unknown}$, $E \subseteq G' \times G'$ are the edges, where each edge $(g_i, g_j) \in E$ reflects an implication for which an assignment of gate g_i leads to the assignment of gate g_j .

Circuit:



Implication graph:

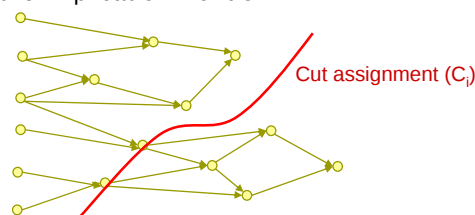


82

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – conflict-based

- The roots (w/o fanin-edges) of the implication graph correspond to the decision vertices, the leaves correspond to the implication frontier



- There is a strict implication order in the graph from the roots to the leaves
 - We can completely cut the graph at any point and identify value assignments to the cut vertices, we result in identical implications toward the leaves

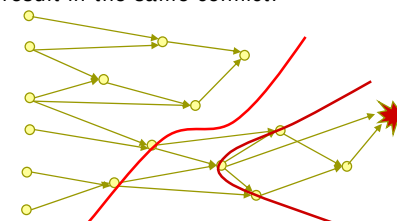
$$C_1 \Rightarrow C_2 \Rightarrow \dots \Rightarrow C_{n-1} \Rightarrow C_n \quad (C_1: \text{decision vertices})$$

83

Boolean Reasoning SAT – AIG-based Decision Procedure

Learning – conflict-based

- If an implication leads to a conflict, any cut assignment in the implication graph between the decision vertices and the conflict will result in the same conflict!



$$(C_i \Rightarrow \text{Conflict}) \Rightarrow (\text{NOT}(\text{Conflict}) \Rightarrow \text{NOT}(C_i))$$

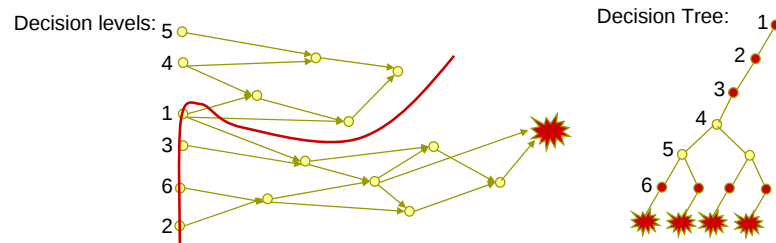
- We can learn the complement of the cut assignment as circuit
 - find minimal cut in the implication graph I (costs less to learn)
 - find dominator vertex if exists
 - restrict size of cuts to be learned, otherwise exponential blow-up

84

Boolean Reasoning SAT – AIG-based Decision Procedure

Non-chronological backtracking

- If we learned only cuts on decision vertices, only the decision vertices that are in the support of the conflict are needed



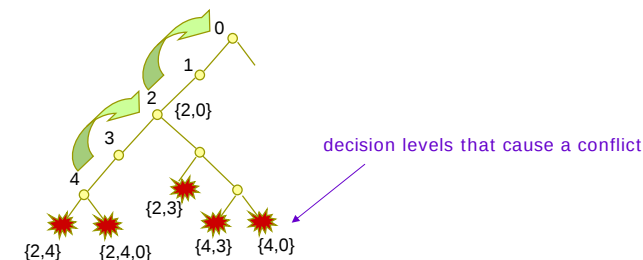
- The conflict is **fully symmetric** with respect to the unrelated decision vertices!!
 - Learning the conflict would prevent checking the symmetric parts again
 - BUT: It is too expensive to learn all conflicts (any cut)

85

Boolean Reasoning SAT – AIG-based Decision Procedure

Non-chronological backtracking

- We can still avoid exploring symmetric parts of the decision tree by tracking the decision support vertices of a conflict
 - If no conflict of the first choice on a decision vertex depends on that vertex, the other choice will result in symmetric conflicts and their evaluation can be skipped!
- By tracking the implications of the decision vertices we can skip decision levels during backtrack



86

Boolean Reasoning SAT – CNF-based Decision Procedure

CNF

- Product-of-Sums (POS) representation of Boolean function
- Describes solution using a set of constraints
 - very handy in many applications because new constraints can be simply added to the list of existing constraints
 - very common in AI community
- Example

$$\varphi = (a+b'+c)(a'+b+c)(a+b'+c')(a+b+c)$$
- SAT on CNF (POS) $\Leftrightarrow$ TAUTOLOGY on DNF (SOP)

87

Boolean Reasoning SAT – CNF-based Decision Procedure

Circuit to CNF conversion

- Encountered often in practical applications
- Naive conversion from circuit to CNF:
 - multiply out expressions of circuit until two level structure
 - Example: $y = x_1 \oplus x_2 \oplus x_3 \oplus \dots \oplus x_n$ (parity function)
 - circuit size is linear in the number of variables



- generated chess-board Karnaugh map
- CNF (or DNF) formula has 2^{n-1} terms (exponential in the # vars)
- Better approach:
 - introduce one variable per circuit vertex
 - formulate the circuit as a conjunction of constraints imposed on the vertex values by the gates
 - uses more variables but size of formula is linear in the size of the circuit

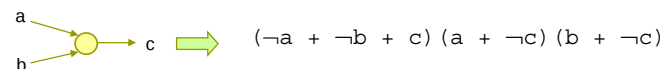
88

Boolean Reasoning SAT – CNF-based Decision Procedure

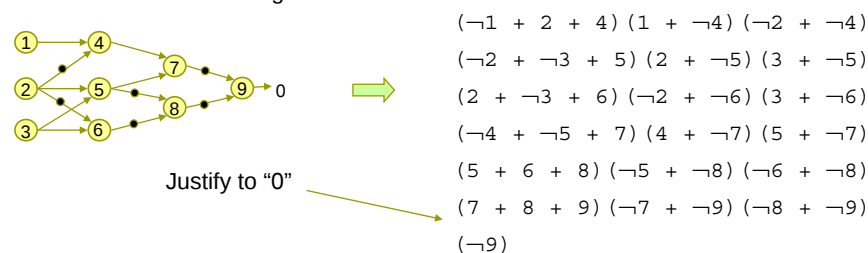
□ Circuit to CNF conversion

■ Example

□ Single gate



□ Connected gates



89

Boolean Reasoning SAT – CNF-based Decision Procedure

□ DPLL procedure

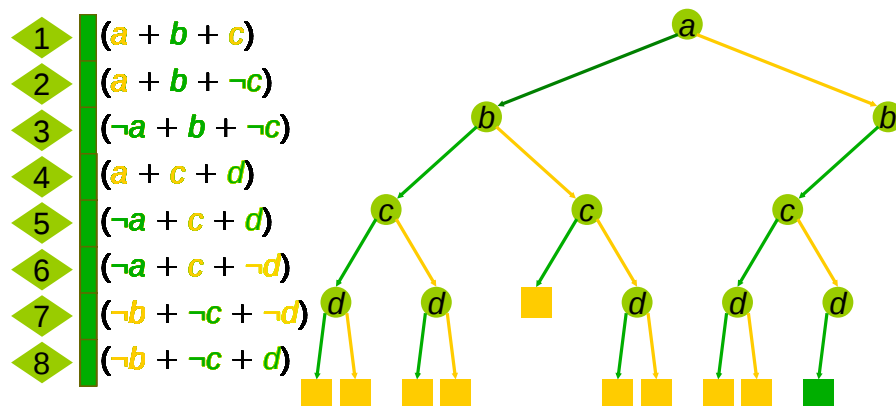
```
Algorithm DPLL() {
  while ChooseNextAssignment() {
    while Deduce() == CONFLICT {
      blevel = AnalyzeConflict();
      if (blevel < 0) return UNSATISFIABLE;
      else Backtrack(blevel);
    }
    return SATISFIABLE;
  }
}
```

ChooseNextAssignment picks next decision variable and assignment
 Deduce does Boolean Constraint Propagation (implications)
 AnalyzeConflict backprocesses from conflict and produces learnt-clause
 Backtrack undoes assignments

90

Boolean Reasoning SAT – CNF-based Decision Procedure

□ DPLL (basic case splitting)



91

Boolean Reasoning SAT – CNF-based Decision Procedure

□ Implication

■ Implications in a CNF formula are caused by unit clauses

- A unit clause is a CNF term for which all variables except one are assigned
 - the value of that clause can be implied immediately

□ Example

clause $(a + \neg b + c)$
 $(a=0) (b=1) \Rightarrow (c=1)$

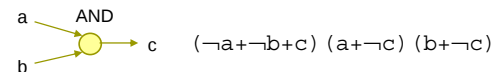
92

Boolean Reasoning

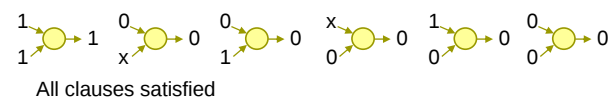
SAT – CNF-based Decision Procedure

Implication

Example



Non-implication cases:



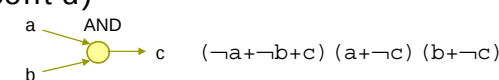
93

Boolean Reasoning

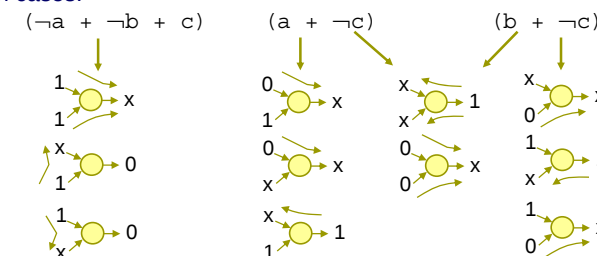
SAT – CNF-based Decision Procedure

Implication

Example (cont'd)



Implication cases:



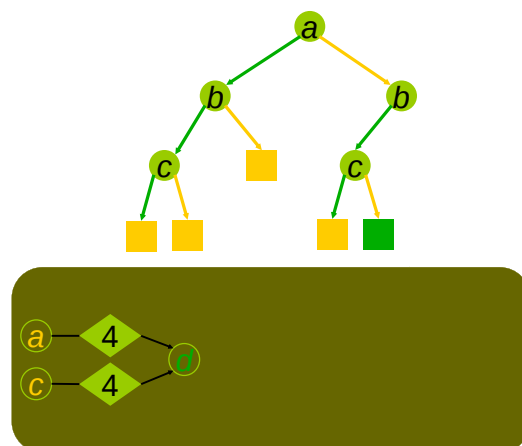
94

Boolean Reasoning

SAT – CNF-based Decision Procedure

DPLL (w/ implication)

- 1 $(a + b + c)$
- 2 $(a + b + \neg c)$
- 3 $(\neg a + b + \neg c)$
- 4 $(a + c + d)$
- 5 $(\neg a + c + d)$
- 6 $(\neg a + c + \neg d)$
- 7 $(\neg b + \neg c + \neg d)$
- 8 $(\neg b + \neg c + d)$



Source: Karem A. Sakallah, Univ. of Michigan

95

Boolean Reasoning

SAT – CNF-based Decision Procedure

Conflict-based learning

Important detail for cut selection:

- During implication processing, record decision level for each implication
- At conflict, select earliest cut such that **exactly one node of the implication graph lies on current decision level**
 - Either decision variable itself
 - Or UIP ("unique implication point") that represents a dominator node for current decision level in conflict graph

- By selecting such cut, implication processing will automatically flip decision variable (or UIP variable) to its complementary value

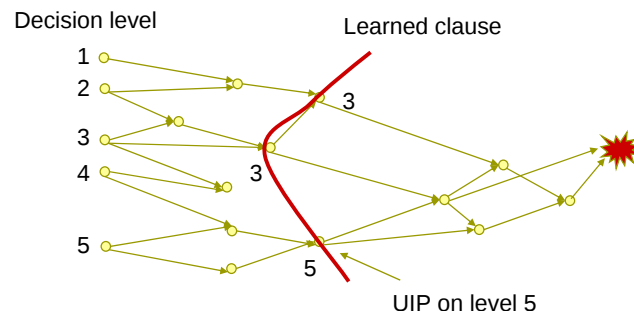
96

Boolean Reasoning SAT – CNF-based Decision Procedure

Conflict-based learning

UIP detection

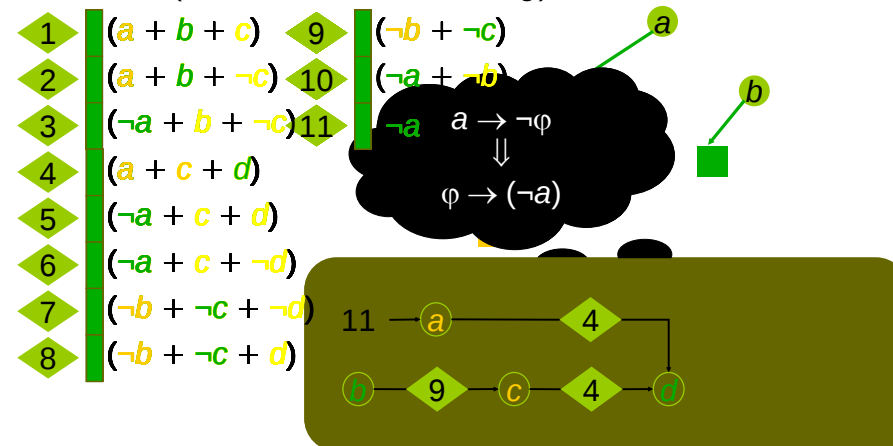
- Store with each implication the decision level, and a time stamp (integer that is incremented after each decision)
 - UIP on decision level l has the property that all following implications towards the conflict have a larger time stamp
 - When back processing from conflict, put all implications that are to be processed on heap, keeping the one with smallest time stamp on top
 - If during processing there is only one variable on current decision level on heap, then that variable must be a UIP



97

Boolean Reasoning SAT – CNF-based Decision Procedure

DPLL (conflict-based learning)



Source: Karem A. Sakallah, Univ. of Michigan

98

Boolean Reasoning SAT – CNF-based Decision Procedure

Implementation issues

- Clauses are stores in arrays
- Track change-sensitive clauses (two-literal watching)
 - all literals but one assigned $\rightarrow$ implication
 - all literals but two assigned $\rightarrow$ clause is sensitive to a change of either literal
 - all other clauses are insensitive and do not need to be observed
- Learning:
 - learned implications are added to the CNF formula as additional clauses
 - limit the size of the clause
 - limit the "lifetime" of a clause, will be removed after some time
- Non-chronological back-tracking
 - similar to circuit case

99

Boolean Reasoning SAT – CNF-based Decision Procedure

Implementation issues (cont'd)

- Random restarts:
 - stop after a given number of backtracks
 - start search again with modified ordering heuristic
 - keep learned structures !
 - very effective for satisfiable formulas, often also effective for unsat formulas
- Learning of equivalence relations:
 - if $(a \Rightarrow b) \wedge (b \Rightarrow a)$, then $(a = b)$
 - very powerful for formal equivalence checking
- Incremental SAT solving
 - solving similar CNF formulas in a row
 - share learned clauses

100