

Logic Synthesis and Verification

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Don't Cares and Node Minimization

Reading:

Logic Synthesis in a Nutshell
Section 3 (§3.4)

part of the following slides are by
courtesy of Andreas Kuehlmann

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Node Minimization

Problem:

- Given a Boolean network, optimize it by minimizing each node as much as possible

Note:

- Assume initial network structure is given
 - Typically obtained after the global optimization, e.g. division and resubstitution
- We minimize the function associated with each node

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Permissible Functions of a Node

- In a Boolean network, we may represent a node using the primary inputs $\{x_1, \dots, x_n\}$ **plus** the intermediate variables $\{y_1, \dots, y_m\}$, as long as the network is **acyclic**

Definition:

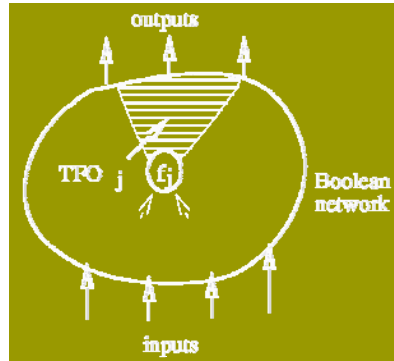
A function g_j , whose input variables are a subset of $\{x_1, \dots, x_n, y_1, \dots, y_m\}$, is **implementable** at a node j if

- the variables of g_j do not intersect with TFO_j
 - $\text{TFO}_j = \{\text{node } i: i = j \text{ or } \exists \text{ path from } j \text{ to } i\}$
- the replacement of the function associated with j , say f_j , by g_j does not change the **functionality** of the network

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Permissible Functions of a Node

- The set of **implementable (or permissible)** functions at j provides the solution space of the local optimization at node j



$$TFO_j = \{\text{node } i: i = j \text{ or } \exists \text{ path from } j \text{ to } i\}$$

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Prime and Irredundant Boolean Network

- Consider a sum-of-products expression F_j associated with a node j
- **Definition:** F_j is **prime** (in a multi-level sense) if for all cubes $c \in F_j$, no **literal** of c can be removed without changing the functionality of the network
- **Definition:** F_j is **irredundant** if for any cube $c \in F_j$, the removal of c from F_j changes the functionality of the network
- **Definition:** A Boolean network is prime and irredundant if F_j is prime and irredundant for all j

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Node Minimization

Goals:

- Given a Boolean network:
 1. make the network prime and irredundant
 2. for a given node of the network, find a **least-cost** SOP expression among the implementable functions at the node

Note:

- Goal 2 implies Goal 1
- There are many expressions that are prime and irredundant, just like in two-level minimization. We seek the **best**.

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Taxonomy of Don't Cares

- **External don't cares** - **XDC**
 - The set of don't care minterms (in terms of primary input variables) given for each primary output is denoted XDC_k , $k=1, \dots, p$
- **Internal don't cares** - derived from the network structure
 - **Satisfiability don't cares** - **SDC**
 - **Observability don't cares** - **ODC**
- **Complete Flexibility** - **CF**
 - CF is a superset of SDC, ODC, and localized XDC

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Satisfiability Don't Cares

- We may represent a node using the n primary inputs plus the m intermediate variables
 - Boolean space is B^{n+m}
- However, intermediate variables depend on the primary inputs
- Thus not all the minterms of B^{n+m} can occur:
 - use the non-occurring minterms as don't cares to optimize the node function
 - we get internal don't cares even when no external don't cares exist

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Satisfiability Don't Cares

Example

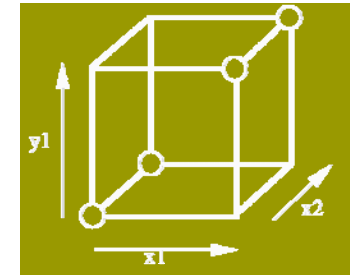
$$y_1 = F_1 = \neg x_1$$

$$y_j = F_j = y_1 x_2$$

- Since $y_1 = \neg x_1$, $y_1 \oplus \neg x_1$ never occurs. So we may include these points to represent F_j

⇒ Don't Cares

$$SDC = (y_1 \oplus \neg x_1) + (y_j \oplus y_1 x_2)$$



In general,

$$SDC = \sum_{j=1}^m (y_j \bar{F}_j + \bar{y}_j F_j)$$

Note: $SDC \subseteq B^{n+m}$

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Observability Don't Cares

$$y_j = \neg x_1 x_2 + x_1 \neg x_3$$

$$z_k = x_1 x_2 + y_j \neg x_2 + \neg y_j \neg x_3$$

- Any minterm of $x_1 x_2 + \neg x_2 \neg x_3 + x_2 x_3$ determines z_k independent of y_j
- The ODC of y_j w.r.t. z_k is the set of minterms of the primary inputs for which the value of y_j is **not observable** at z_k

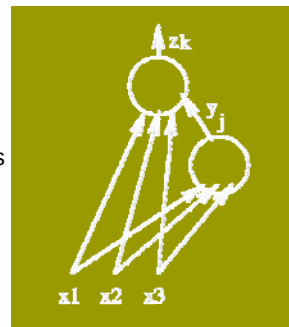
$$ODC_{jk} = \{x \in B^n \mid z_k(x)|_{y_j=0} \equiv z_k(x)|_{y_j=1}\}$$

This means that the two Boolean networks,

- one with y_j forced to 0 and
- one with y_j forced to 1

compute the same value for z_k when $x \in ODC_{jk}$

- The ODC of y_j w.r.t. all primary outputs is $ODC_j = \cap_k ODC_{jk}$



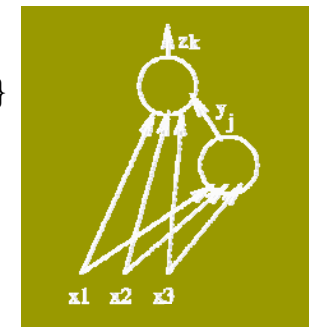
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Observability Don't Cares

$$ODC_{jk} = \{x \in B^n \mid z_k(x)|_{y_j=0} = z_k(x)|_{y_j=1}\}$$

$$\text{denote } ODC_{jk} = \frac{\partial z_k}{\partial y_j}$$

$$\text{where } \frac{\partial z_k}{\partial y_j} = z_k(x)|_{y_j=0} \oplus z_k(x)|_{y_j=1}$$



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Observability Don't Cares

- The ODCs of node i and node j in a Boolean network may not be compatible
 - Modifying the function of node i using ODC_i may invalidate ODC_j
 - It brings up the issue of **compatibility** ODC (CODC)
 - Computing CODC is too expensive to be practical
 - Practical approaches to node minimization often consider one node at a time rather than multiple nodes simultaneously

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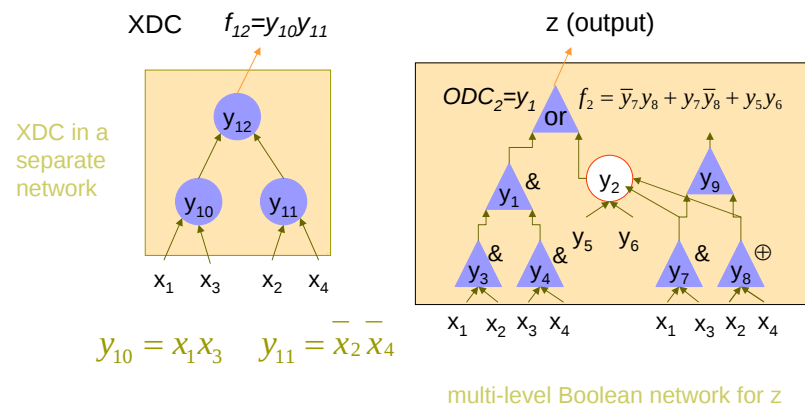
External Don't Cares

- The XDC global for an entire Boolean network is often given
- The XDC local for a specified window in a Boolean network can be computed
- Question:
 - How do we represent XDC?
 - How do we translate XDC into local don't care?
 - XDC is originally in PI variables
 - Translate XDC in terms of input variables of a node

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External Don't Cares

Representing XDC

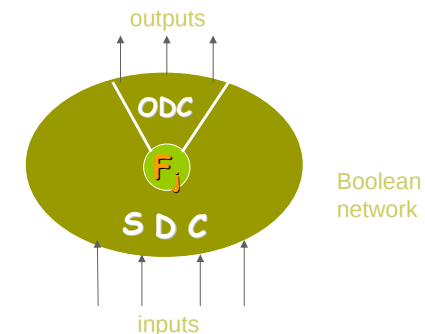


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Don't Cares of a Node

- The don't cares of a node j can be computed by

$$DC_j = \sum_{i \in TFO_j} (y_i \bar{F}_i + \bar{y}_i F_i) + \prod_{k=1}^p (ODC_{jk} + XDC_k)$$



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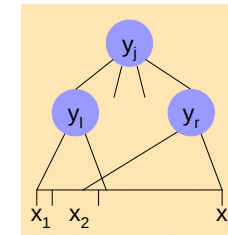
Don't Cares of a Node

- **Theorem:** The function $\mathcal{F}_j = (F_j - DC_j, DC_j, \neg(F_j + DC_j))$ is the complete set of implementable functions at node j
- **Corollary:** F_j is prime and irredundant (in the multi-level sense) iff it is prime and irredundant cover of $\mathcal{F}_j$
- A least-cost expression at node j can be obtained by minimizing $\mathcal{F}_j$
- A prime and irredundant Boolean network can be obtained by using only 2-level logic minimization for each node j with the don't care DC_j

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Mapping Don't Cares to Local Space

- How can **ODC** + **XDC** be used for optimizing a node j ?
 - ODC and XDC are in terms of the primary input variables
 - Need to convert to the input variables of node j



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Mapping Don't Cares to Local Space

- **Definition:** The **local space** B^r of node j is the Boolean space spanned by the fanin variables of node j (plus maybe some other variables chosen selectively)
 - A don't care set $D(y^{r+})$ computed in local space spanned by y^{r+} is called a local don't care set. (The "+" stands for additional variables.)
 - **Solution:** Map $DC(x) = ODC(x) + XDC(x)$ to local space of the node to find local don't cares, i.e.,

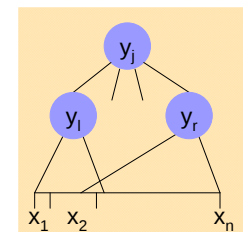
$$D(y^{r+}) = \overline{IMG_{g_{FI_j^+}}(DC(x))}$$

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Mapping Don't Cares to Local Space

- Computation in two steps:
 1. Find $DC(x)$ in terms of primary inputs
 2. Find D , the local don't care set, by image computation and complementation

$$D(y^{r+}) = \overline{IMG_{g_{FI_j^+}}(DC(x))}$$

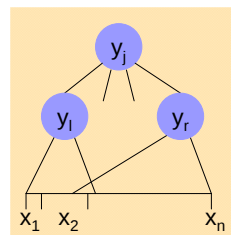


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Mapping Don't Cares to Local Space Global Function of a Node

$$y_j = \begin{cases} f_j(y_k, \dots, y_l) \\ g_j(x_1, \dots, x_n) \end{cases} \text{ global function}$$

$$B^{m+n} \rightarrow B^n$$



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Mapping Don't Cares to Local Space Don't Cares in Primary Inputs

□ BDD based computation

- Build BDD's representing global functions at each node
 - in both the primary network and the don't care network, $g_j(x_1, \dots, x_n)$
 - use `BDD_compose`
- Replace all the intermediate variables in (ODC+XDC) with their global BDDs

$$\tilde{h}(x, y) = DC(x, y) \rightarrow h(x) = DC(x)$$

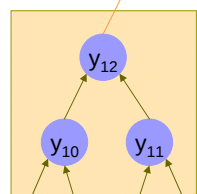
$$\tilde{h}(x, y) = \tilde{h}(x, g(x)) = h(x)$$

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Mapping Don't Cares to Local Space

□ Example

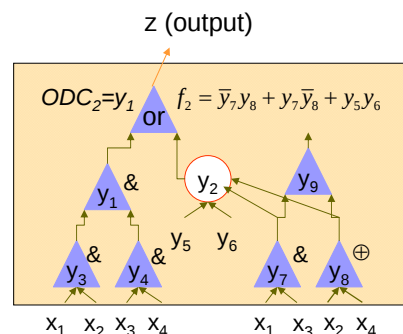
$$XDC \quad f_{12} = y_{10}y_{11}$$



$$y_{10} = x_1x_3 \quad y_{11} = \bar{x}_2\bar{x}_4$$

$$XDC_2 = y_{12}$$

$$g_{12} = x_1x_2x_3x_4$$



$$ODC_2 = y_1$$

$$g_1 = x_1x_2x_3x_4$$

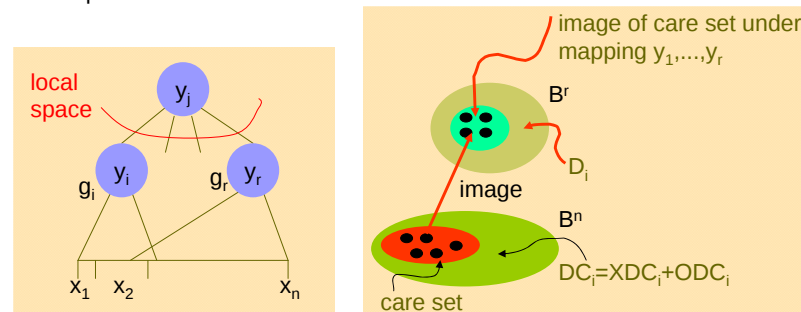
$$DC_2 = ODC_2 + XDC_z$$

$$DC_2 = x_1x_2x_3x_4 + \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$$

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Mapping Don't Cares to Local Space Image Computation

- Local don't cares are the set of minterms in the local space of y_i that cannot be reached under any input combination in the **care set** of y_i (in terms of the input variables).
- Local don't care set: $D_i = \overline{\text{IMAGE}_{(g_i, g_2, \dots, g_r)}[\text{care set}]}$
i.e. those patterns of $(y_1, \dots, y_r)$ that never appear as images of input cares.

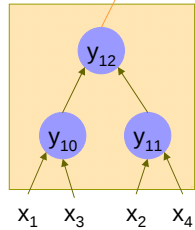


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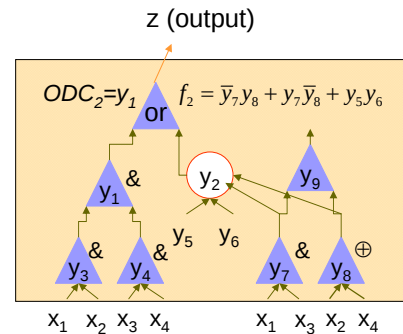
Mapping Don't Cares to Local Space

Example (cont'd)

XDC $f_{12} = y_{10}y_{11}$



$ODC_2 = y_1$
 $ODC_4 = y_{12}$
 $DC_2 = x_1x_2x_3x_4 + x_1x_2x_3x_4$
 $DC_4 = x_1 + x_3 + x_2x_4 + x_2x_4$
 $D_2 = y_7y_8$



Note that D_2 is given in this space y_5, y_6, y_7, y_8 .
 Thus in the space $(- - 10)$ never occurs.
 Can check that $DC_2D_2 = \emptyset = DC_2(x_1x_3)(x_2x_4 + x_2x_4)$
 Using $D_2 = y_7y_8$, f_2 can be simplified to
 $f_2 = y_7y_8 + y_5y_6$

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Image Computation

Two methods:

1. Transition relation method

$f : B^n \rightarrow B^r \Rightarrow F : B^n \times B^r \rightarrow B$
 (F is the characteristic function of f!)

$$F(x, y) = \{(x, y) \mid y = f(x)\}$$

$$= \prod_{i \leq r} (y_i \equiv f_i(x))$$

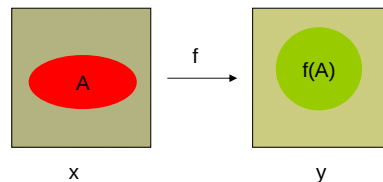
$$= \prod_{i \leq r} (y_i f_i(x) + \bar{y}_i \bar{f}_i(x))$$

2. Recursive image computation (omitted)

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Image Computation Transition Relation Method

Image of set A under f: $f(A) = \exists_x (F(x, y) \wedge A(x))$



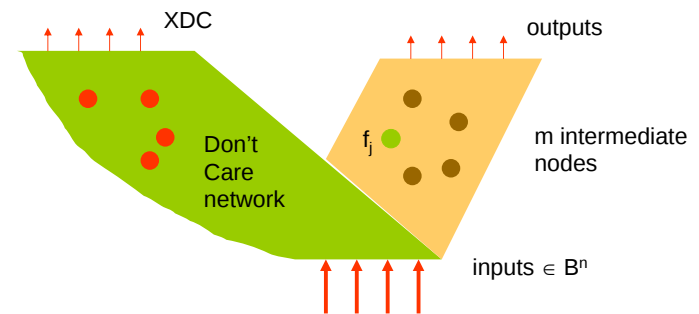
where $\exists_x = \exists_{x_1} \dots \exists_{x_n}$ and $\exists_{x_i} g = g_{x_i} + g_{\bar{x}_i}$

The existential quantification $\exists_x$ is also called "smoothing"

Note: The result is a BDD representing the image,
 i.e. $f(A)$ is a BDD with the property that
 $BDD(y) = 1 \Leftrightarrow \exists x \text{ such that } f(x) = y \text{ and } x \in A.$

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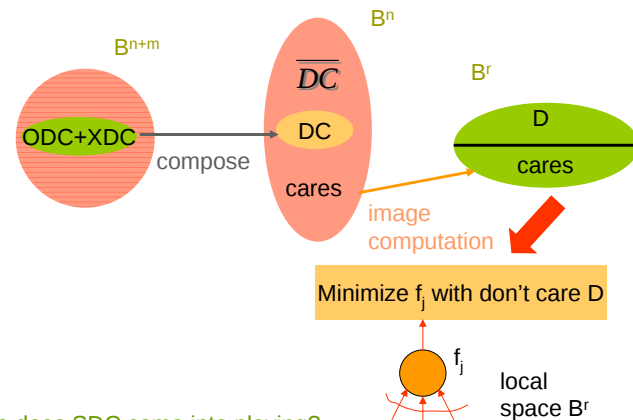
Node Simplification



Express ODC in terms of variables in B^{n+m}

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Node Simplification



Question: Where does SDC come into playing?

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Complete Flexibility

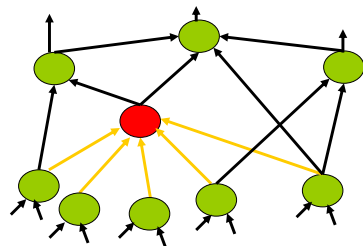
Complete flexibility (CF) of a node in a combinational network

- SDC + ODC + localized XDC
- Used to minimize one node at a time
 - Not considering compatible flexibilities among multiple nodes
 - Different from CODC, where don't cares at different nodes are compatible and can minimize multiple nodes simultaneously

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Complete Flexibility

- Definition: A **flexibility** at a node is a *relation* (between the node's inputs and output) such that *any well-defined sub-relation used at the node* leads to a network that conforms to the external specification
- Definition: The **complete flexibility (CF)** is the *maximum* flexibility possible at a node

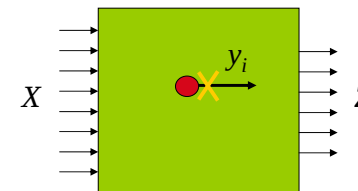


Combinational Logic Network

by courtesy of Robert Brayton 31

Complete Flexibility

Computing complete flexibility



$$I(X, y_i, Z)$$

cut the network and treat y_i as a pseudo primary input

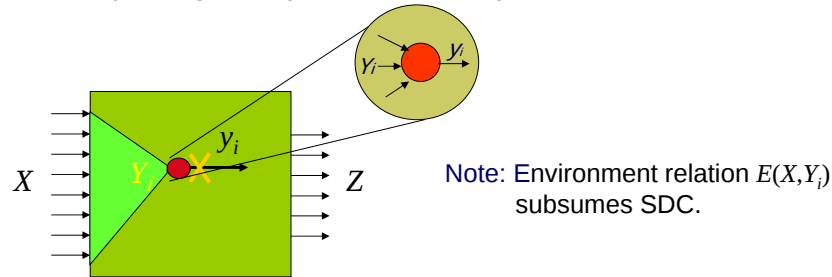
$$R(X, y_i) = \forall Z. [I(X, y_i, Z) \Rightarrow S(X, Z)]$$

Note: Specification relation $S(X, Z)$ may contain non-determinism and subsumes XDC. Influence relation $I(X, y_i, Z)$ subsumes ODC.

by courtesy of Robert Brayton 32

Complete Flexibility

Computing complete flexibility

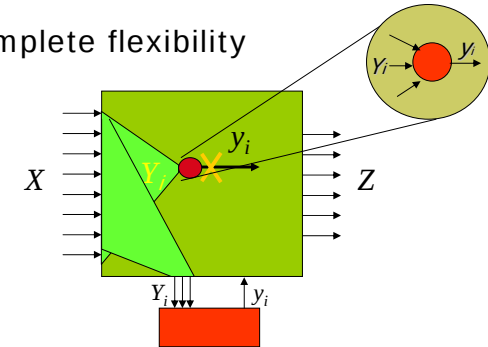


$$\begin{aligned}
 CF(Y_i, y_i) &= \forall X. [E(X, Y_i) \Rightarrow R(X, y_i)] \\
 &= \forall X. [E(X, Y_i) \Rightarrow \forall Z. [I(X, y_i, Z) \Rightarrow S(X, Z)]] \\
 &= \forall X, Z. \neg [E(X, Y_i) \wedge I(X, y_i, Z) \wedge \neg S(X, Z)]
 \end{aligned}$$

by courtesy of Robert Brayton 33

Complete Flexibility

Computing complete flexibility



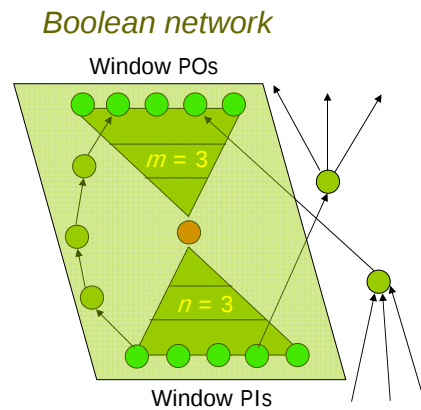
$$\begin{aligned}
 CF(Y_i, y_i) &= \forall X. [E(X, Y_i) \Rightarrow \forall Z. [I(X, y_i, Z) \Rightarrow S(X, Z)]] \\
 &= \forall X, Z. \overline{E(X, Y_i) \cdot I(X, y_i, Z) \cdot \overline{S(X, Z)}}
 \end{aligned}$$

Note: The same computation works for multiple y_i 's

by courtesy of Robert Brayton 34

Window and Don't Care Computation

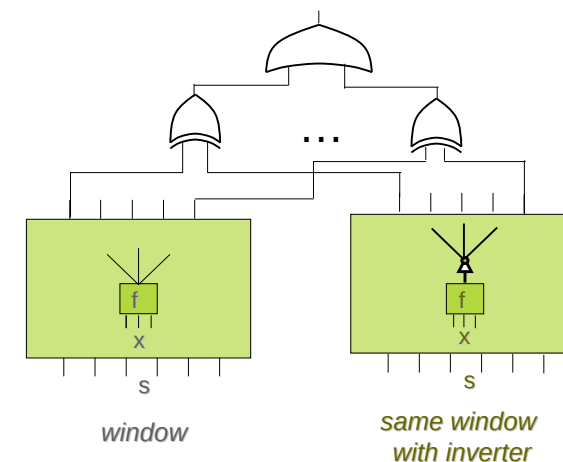
- Definition: A **window** for a node in the network is the context in which the don't-cares are computed
- A window includes
 - n levels of the TFI
 - m levels of the TFO
 - all re-convergent paths captured in this scope
- Window with its PIs and POs can be considered as a separate network
- Optimizing a window is more computationally affordable than optimizing an entire network



by courtesy of Alan Mishchenko 35

SAT-based Don't Care Computation

"Miter" constructed for the window POs



by courtesy of Alan Mishchenko 36

SAT-based Don't Care Computation

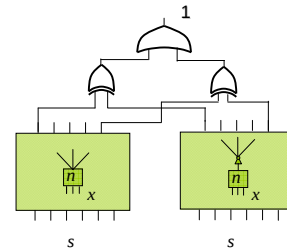
□ Compute the care set

■ Simulation

- Simulate the miter using random patterns
- Collect x minterms, for which the output of miter is 1
- This is a subset of a care set

■ Satisfiability

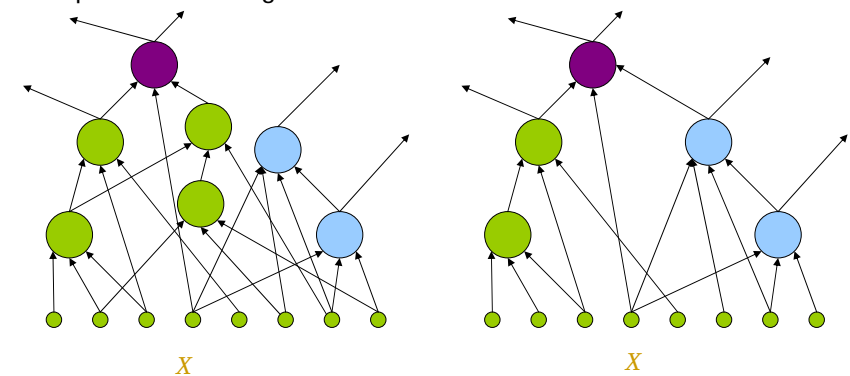
- Derive set of network clauses
- Add the negation of the current care set
- Assert the output of miter to be 1
- Enumerate through the SAT assignments
- Add these assignments to the care set



by courtesy of Alan Mishchenko 37

Resubstitution for Circuit Minimization

- Resubstitution considers a node in a Boolean network and expresses it using a different set of fanins



Computation can be enhanced by use of don't cares

by courtesy of Alan Mishchenko 38

Resubstitution with Don't Cares

□ Consider all or some nodes in Boolean network

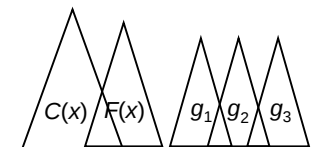
- Create window
- Select possible fanin nodes (divisors)
- For each candidate subset of divisors
 - Rule out some subsets using simulation
 - Check resubstitution feasibility using SAT
 - Compute resubstitution function using interpolation
 - A low-cost by-product of completed SAT proofs
- Update the network if there is an improvement

by courtesy of Alan Mishchenko 39

Resubstitution with Don't Cares

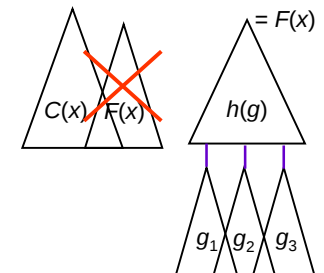
□ Given:

- node function $F(x)$ to be replaced
- care set $C(x)$ for the node
- candidate set of divisors $\{g_i(x)\}$ for re-expressing $F(x)$



□ Find:

- A resubstitution function $h(y)$ such that $F(x) = h(g(x))$ on the care set
- Necessary and sufficient condition:
For any minterms a and b , $F(a) \neq F(b)$ implies $g_i(a) \neq g_i(b)$ for some g_i



by courtesy of Alan Mishchenko 40

Resubstitution

Example

Given:

$$F(x) = (x_1 \oplus x_2)(x_2 \vee x_3)$$

Two candidate sets:

$$\{g_1 = x_1 \cdot x_2, g_2 = x_1 \cdot x_2 \cdot x_3\},$$

$$\{g_3 = x_1 \vee x_2, g_4 = x_2 \cdot x_3\}$$

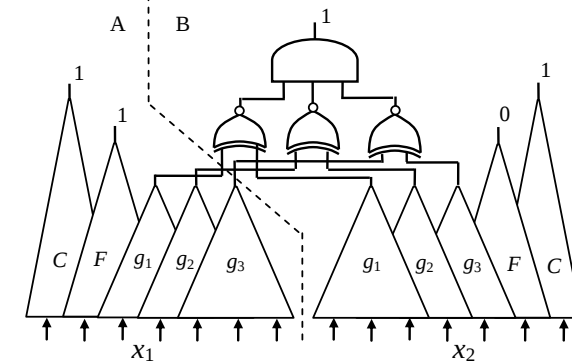
Set $\{g_3, g_4\}$ cannot be used for resubstitution while set $\{g_1, g_2\}$ can.

x	F(x)	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$
000	0	0	0	0	0
001	0	0	0	0	0
010	1	1	0	1	0
011	1	1	0	1	1
100	0	0	0	1	0
101	1	0	1	1	0
110	0	0	0	1	0
111	0	0	0	1	1

by courtesy of Alan Mishchenko 41

SAT-based Resubstitution

Miter for resubstitution check



Resubstitution function exists if and only if SAT problem is unsatisfiable

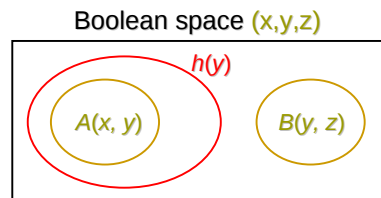
Note: Care set is used to enhance resubstitution check

by courtesy of Alan Mishchenko 42

SAT-based Resubstitution

Computing dependency function h by interpolation

- Consider two sets of clauses, $A(x, y)$ and $B(y, z)$, such that $A(x, y) \wedge B(y, z) = 0$
- y are the only variables common to A and B
- An **interpolant** of the pair $(A(x, y), B(y, z))$ is a function $h(y)$ depending only on the common variables y such that $A(x, y) \Rightarrow h(y) \Rightarrow \neg B(y, z)$



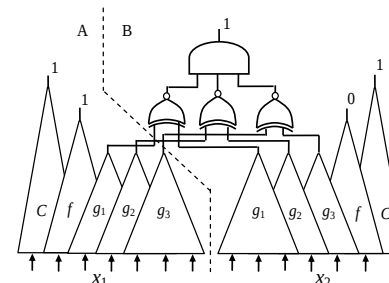
by courtesy of Alan Mishchenko 43

SAT-based Resubstitution

Problem: Find function $h(y)$, such that $C(x) \Rightarrow [h(g(x)) \equiv F(x)]$, i.e. $F(x)$ is expressed in terms of $\{g_i\}$

Solution:

- Prove the corresponding SAT problem "unsatisfiable"
- Derive unsatisfiability resolution proof [Goldberg/Novikov, DATE'03]
- Divide clauses into A clauses and B clauses
- Derive interpolant from the unsatisfiability proof [McMillan, CAV'03]
- Use interpolant as the dependency function, $h(g)$
- Replace $F(x)$ by $h(g)$ if cost function improved



by courtesy of Alan Mishchenko 44