

## Problem Set 2 Solution

### 1 [Cofactor and QBF]

(a)

$$\begin{aligned}
 f &= v \cdot f_v + \neg v \cdot f_{\neg v} \\
 \neg f &= \neg(v \cdot f_v + \neg v \cdot f_{\neg v}) \\
 &= \neg(v \cdot f_v) \cdot \neg(\neg v \cdot f_{\neg v}) \\
 &= (\neg v + \neg f_v) \cdot (v + \neg f_{\neg v}) \\
 &= v \cdot \neg f_v + \neg v \cdot \neg f_{\neg v} + \neg f_v \cdot \neg f_{\neg v} \\
 &= v \cdot (\neg f_v + \neg f_{\neg v}) + \neg v \cdot (\neg f_{\neg v} + \neg f_v) \\
 &= v \cdot (\neg f_v) + \neg v \cdot (\neg f_{\neg v}) \\
 &= v \cdot (\neg f)_v + \neg v \cdot (\neg f)_{\neg v} \quad \xrightarrow{\text{shannon expansion of } \neg f} \\
 \Rightarrow (\neg f)_v &= \neg f_v \quad \text{#}
 \end{aligned}$$

$$\begin{aligned}
 f \oplus g &= f \cdot g' + f' \cdot g \\
 &= (v \cdot f_v + v' \cdot f_{\neg v}) \cdot (v \cdot g'_v + v' \cdot g'_{\neg v}) + (v \cdot f'_v + v' \cdot f'_{\neg v}) \cdot (v \cdot g_v + v' \cdot g_{\neg v}) \\
 &= [v(f_v g'_v) + v \cdot (f_v \cdot g'_{\neg v})] + [v \cdot (f'_v g_v) + v' \cdot (f'_v \cdot g_{\neg v})] \\
 &= v \cdot (f_v g'_v + f'_v g_v) + v' \cdot (f_v \cdot g'_{\neg v} + f'_v \cdot g_{\neg v}) \\
 \Rightarrow (f \oplus g)_v &= f_v \cdot g'_v + f'_v \cdot g_v \quad \text{--- ①} \\
 f_v \oplus g_v &= f_v \cdot g'_v + f'_v \cdot g_v \quad \text{--- ②} \\
 \text{②} &= \text{①} \quad \therefore \text{#}
 \end{aligned}$$

(b) if  $a = b \Rightarrow a \rightarrow b \wedge b \rightarrow a \Rightarrow a \leftrightarrow b$

$$\begin{aligned}
 \forall x \exists y \ f(x, y, z) &= \forall x f_y \vee f_{\neg y} = (f_{\neg x y} \vee f_{x y}) \wedge (f_{\neg x y} \vee f_{x \neg y}) \\
 &= (f_{\neg x y} \wedge f_{x y}) \vee (f_{\neg x y} \wedge f_{x \neg y}) \vee (f_{x \neg y} \wedge f_{x y}) \\
 \exists y \forall x \ f(x, y, z) &= \exists y (f_x \wedge f_{\neg x}) = (f_{\neg x y} \wedge f_{x y}) \vee (f_{x \neg y} \wedge f_{x y}) \\
 &\sim \text{counter example: } f = \bar{x}y + xy \\
 \forall x \exists y \ f &= \forall x (\bar{x} + x) = \exists y \forall x f = \exists y (y \cdot y) = 0 \\
 \textcircled{①} \forall x \exists y \ f(x, y, z) &= [(\neg f_{\bar{x} y}) \wedge (\neg f_{x y})] \vee [(\neg f_{x \bar{y}}) \wedge (\neg f_{x y})] \\
 \exists x (\neg \exists y \ f(x, y, z)) &= \exists x [(\neg f_{\bar{x} y}) \wedge (\neg f_{x y})] = [(\neg f_{\bar{x} y}) \wedge (\neg f_{x y})] \vee \\
 &\quad [(\neg f_{x \bar{y}}) \wedge (\neg f_{x y})] = \neg \forall x \exists y f
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{②} \exists x (f(x, y) \wedge g(x, y)) &= (f_{\bar{x}} \wedge g_{\bar{x}}) \vee (f_{\bar{x}} \wedge g_x) = \\
 \exists x f(x, y) \wedge \exists x g(x, y) &= (f_{\bar{x}} \vee f_x) \wedge (g_{\bar{x}} \vee g_x) = \frac{f_{\bar{x}} \wedge g_{\bar{x}}}{f_x \wedge g_{\bar{x}}} \vee \frac{f_{\bar{x}} \wedge g_x}{f_x \wedge g_x} \vee \frac{f_x \wedge g_{\bar{x}}}{f_x \wedge g_x} \\
 \text{counter example: } f &= \bar{x}, g = x \\
 \exists x (f \wedge g) &= 0, \quad \exists x f \wedge \exists x g = 1
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{④} \exists x (f(x, y) \vee g(x, y)) &= (f_{\bar{x}} \vee g_{\bar{x}}) \vee (f_x \vee g_x) = f_{\bar{x}} \vee f_x \vee g_{\bar{x}} \vee g_x \\
 (\exists x f(x, y)) \vee (\exists x g(x, y)) &= (f_{\bar{x}} \vee g_{\bar{x}}) \vee (g_{\bar{x}} \vee g_x) = f_{\bar{x}} \vee f_x \vee g_{\bar{x}} \vee g_x \\
 &= \exists x (f(x, y) \vee g(x, y))
 \end{aligned}$$

(c)

$$\begin{aligned} \exists z. f(x,y,z) &= f(x,y,1) + f(x,y,0) \\ f(x,y,g(x,y)) &= g(x,y) \cdot f(x,y,1) + g'(x,y) \cdot f(x,y,0) \\ f(x,y,1) + f(x,y,0) &= g(x,y) \cdot f(x,y,1) + g'(x,y) \cdot f(x,y,0) \\ \Rightarrow f(x,y,1) = 0, f(x,y,0) &= 0 \rightarrow g(y,y) = X, g'(x,y) = X \\ 0 &\quad 1 \rightarrow \quad 0 \quad 1 \\ 1 &\quad 0 \rightarrow \quad 1 \quad 0 \\ 1 &\quad 1 \rightarrow \quad X \quad X \end{aligned}$$

$\therefore$  onset of  $g(x,y)$ :  $f_g(x,y) \cdot \bar{f}_{\bar{g}}(x,y)$   
 offset of  $g(x,y)$ :  $\bar{f}_g(x,y) \cdot f_{\bar{g}}(x,y)$   
 don't-care set:  $\bar{f}_g(x,y) \cdot \bar{f}_{\bar{g}}(x,y) + f_g(x,y) \cdot \bar{f}_{\bar{g}}(y,x)$

## 2 [BDD Procedures]

(a) COFACTOR( $F, l$ ) {

```

if (TOP_VARIABLE(F) == l.variable) {
  if (l.sign == 1) return F.l-child
  if (l.sign == 0) return F.0-child
}
i = COFACTOR(F.0-child, l)
e = COFACTOR(F.l-child, l)
return ITE(TOP_VARIABLE(F), i, e)
  }
```

(b)

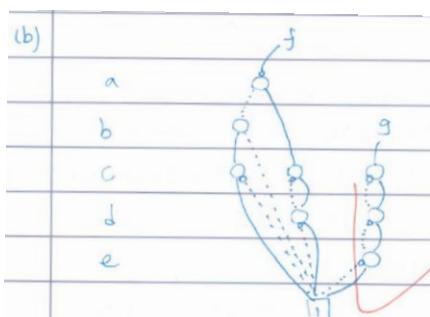
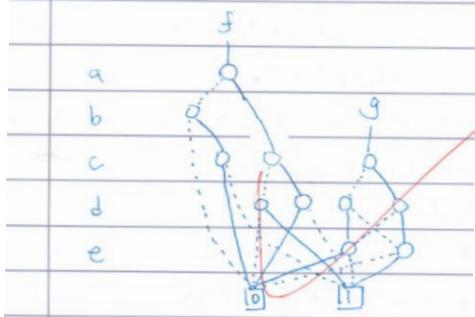
COMPOSE( $F.v, G$ ) {

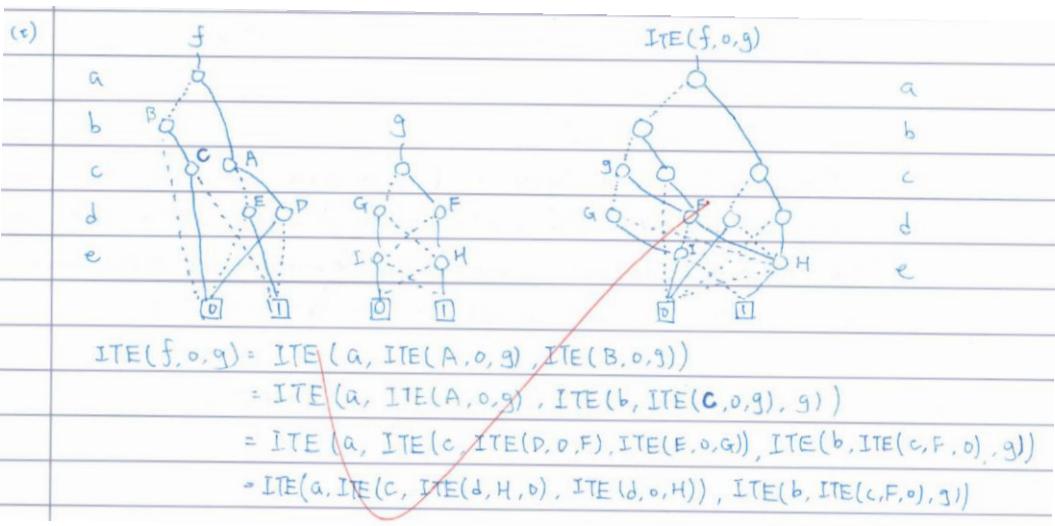
```

if (TOP_VARIABLE(F) > v) return F
if (TOP_VARIABLE(F) == v)
  return ITE(G, COFACTOR(F.v), COFACTOR(F. $\neg v$ ))
i = COMPOSE(COFACTOR(F, TOP_VARIABLE(F)), v, G)
e = COMPOSE(COFACTOR(F,  $\neg$ TOP_VARIABLE(F)), v, G)
return ITE(TOP_VARIABLE(F), i, e)
  }
```

## 3 [BDD Operations]

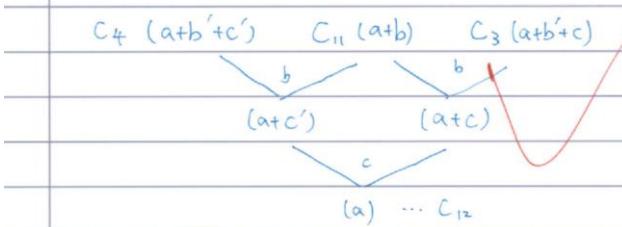
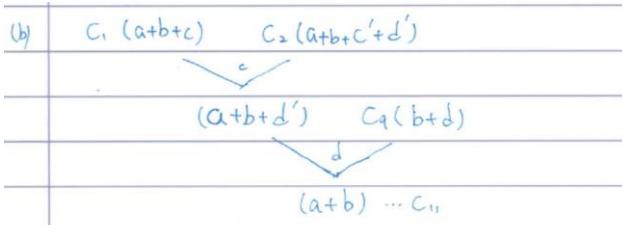
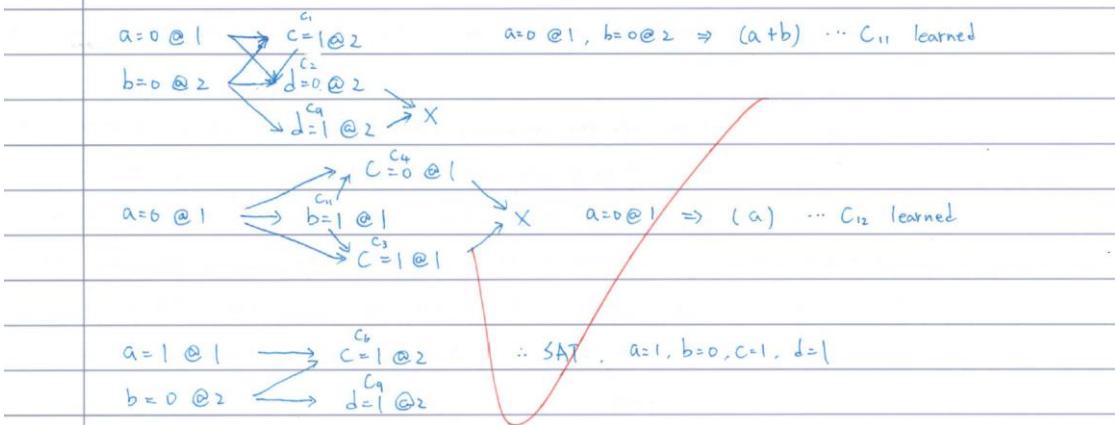
(a)  $f = a'b'c + ac'd + acd'$ ,  $g = c \oplus d \oplus e$





#### 4 [SAT Solving]

4. (a)  $C_1(a+b+c)$ ,  $C_2(a+b+c'+d')$ ,  $C_3(a+b'c)$ ,  $C_4(a+b'c')$ ,  $C_5(a+c'+d)$ ,  $C_6(a'b+c)$   
 $C_7(a'b'd)$ ,  $C_8(a'b'+c'+d')$ ,  $C_9(b+d)$ ,  $C_{10}(b'+c+d')$



## 5 [SAT Solving]

(1) If resolution obtains empty clause , then one of the two parent clauses must be false .

\* ie.  $(x) \ (\neg x)$   $\therefore x$  and  $\neg x$  can't be satisfied at the same time

And for that falsified clause , one of its parent clause would still be false under the same way . So there is at least one clause in  $\phi$  is falsified , and the CNF is UNSAT.

(2) If resolution obtains only one nonempty clause , we can find an assignment to make the clause true . Using this assignment , at least one of the parent clause is true .

For the other clause which may still not be satisfied yet , we can always use the resolved variable to satisfy that clause . In this way , we can find an assignment that satisfies the original CNF in the end , so the CNF is SAT .

\* ie.  $C_1(x+y) \ C_2(\neg x+z)$  if  $x=1 \rightarrow C_1=1$  , we can assign  $x=0$  to make  $C_2=1$

According to (1) and (2) , a CNF formula is UNSAT iff an empty clause can be obtained through resolution .