Proofs and Types
Sums in Natural Deduction

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Hypothesis and Introduction Rules

- *Hypothesis*: A
- *Introductions*:

$$\dfrac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\ \wedge\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\ \vee 1\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\ \vee 2\mathcal{I}$$

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\ \Rightarrow \mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall \xi . A}\ \forall\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A[a/\xi]\end{matrix}}{\exists \xi . A}\ \exists\mathcal{I}$$

  - In $\forall\mathcal{I}$, ξ is not free in any hypothesis.

Elimination Rules

- *Eliminations*:

$$\frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A}\ \wedge 1\mathcal{E} \qquad \frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{B}\ \wedge 2\mathcal{E} \qquad \frac{\begin{array}{c}\vdots\\ \bot\end{array}}{C}\ \bot\mathcal{E}$$

$$\frac{\begin{array}{c}\\ \vdots\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\ \vee\mathcal{E} \qquad \frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\ \Rightarrow \mathcal{E}$$

$$\frac{\begin{array}{c}\vdots\\ \forall\xi.A\end{array}}{A[a/\xi]}\ \forall\mathcal{E} \qquad \frac{\begin{array}{c}\\ \vdots\\ \exists\xi.A\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C}\ \exists\mathcal{E}$$

- ξ must not be free in the hypotheses or the conclusion after using $\exists\mathcal{E}$

Defects

- The introduction rules $\vee 1\mathcal{I}$, $\vee 2\mathcal{I}$, and $\exists\mathcal{I}$ are nice.
  - They are symmetric to $\wedge 1\mathcal{E}$, $\wedge 2\mathcal{E}$, and $\forall\mathcal{E}$ respectively.
- The elimination rules $\perp\mathcal{E}$, $\vee\mathcal{E}$, and $\exists\mathcal{E}$ are bad.
  - They are not symmetric.
  - The formula C comes out of nowhere.
- They also introduce more deductions to the same "proof."

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\qquad\begin{matrix}[A]\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\;\vee\mathcal{E}}{D}\;\mathsf{r} \qquad \text{“equal”} \qquad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\qquad\begin{matrix}[A]\\ \vdots\\ \dfrac{C}{D}\;\mathsf{r}\end{matrix}\qquad\begin{matrix}[B]\\ \vdots\\ \dfrac{C}{D}\;\mathsf{r}\end{matrix}}{D}\;\vee\mathcal{E}$$

Standard Conversions

- New conversions are needed for new rules:
  - $$\dfrac{\dfrac{\begin{array}{c}\vdots \\ A\end{array}}{A \vee B}\,\vee 1\mathcal{I} \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C}\,\vee\mathcal{E} \qquad \text{converts to} \qquad \begin{array}{c}\vdots \\ A \\ \vdots \\ C\end{array}$$
  - $$\dfrac{\dfrac{\begin{array}{c}\vdots \\ B\end{array}}{A \vee B}\,\vee 1\mathcal{I} \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C}\,\vee\mathcal{E} \qquad \text{converts to} \qquad \begin{array}{c}\vdots \\ B \\ \vdots \\ C\end{array}$$
  - $$\dfrac{\dfrac{\begin{array}{c}\vdots \\ A[a/\xi]\end{array}}{\exists\xi.A}\,\exists\mathcal{I} \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array}}{C}\,\exists\mathcal{E} \qquad \text{converts to} \qquad \begin{array}{c}\vdots \\ A[a/\xi] \\ \vdots \\ C\end{array}$$

Principal Premise

- Not every introduction followed by elimination is a redex. Consider

$$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I} \qquad \begin{array}{c}\vdots\\ (A \Rightarrow B) \Rightarrow C\end{array}}{C}\Rightarrow\mathcal{E}$$

- For elimination rules with multiple premises ($\Rightarrow\mathcal{E}$, $\vee\mathcal{E}$, $\exists\mathcal{E}$), a redex has an introduction ending in the *principal* premise.

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\Rightarrow\mathcal{E} \qquad \dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\vee\mathcal{E} \qquad \dfrac{\begin{array}{c}\vdots\\ \exists\xi.A\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C}\exists\mathcal{E}$$

- A *principal branch* of a deduction is a sequence of formulae $A_0, A_1, \ldots, A_n$ that
  - A_0 is an (undischarged) hypothesis;
  - A_n is the conclusion;

Subformula Property

Theorem 1

Let δ be a normal deduction in the $(\wedge, \Rightarrow, \forall)$ fragment. Then

- *every formula in δ is a subformula of the conclusion or a hypothesis of δ;*
- *if δ ends in an elimination, it has a principal branch. (particularly, the conclusion is a subformula of a hypothesis.)*

Proof.

- If δ is a hypothesis, trivil.
- If δ ends in an introduction, the premises are subformulae of the conclusion. IH gives the result. For example,
  $$\frac{A \qquad B}{A \wedge B} \wedge\mathcal{I}.$$
- If δ ends in an elimination, then the proof above the principal premise is not an introduction for δ is normal

Subformula Property

- For the full fragment, the subformula property does not hold.
- The "bad" eliminations can have an arbitrary C.
- Here is a concrete example:

$$\dfrac{\dfrac{A \vee A \qquad \dfrac{[A] \qquad [A]}{A \wedge A}\wedge\mathcal{I} \qquad \dfrac{[A] \qquad [A]}{A \wedge A}\wedge\mathcal{I}}{A \wedge A}\vee\mathcal{E}}{A}\wedge 1\mathcal{E}$$

- Observe that two consecutive eliminations can be exchanged without changing the nature of the "proof."

$$\dfrac{A \vee A \qquad \dfrac{\dfrac{[A] \qquad [A]}{A \wedge A}\wedge\mathcal{I}}{A}\wedge 1\mathcal{E} \qquad \dfrac{\dfrac{[A] \qquad [A]}{A \wedge A}\wedge\mathcal{I}}{A}\wedge 1\mathcal{E}}{A}\vee\mathcal{E}$$

- More conversions are needed!

Commuting Conversions

- $\dfrac{C \qquad \vdots}{D}\,r$ is an elimination of principal premise C with conclusion D.
- Commutation of $\bot\mathcal{E}$.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{C}\,\bot\mathcal{E} \qquad \vdots}{D}\,r \qquad \text{converts to} \qquad \dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{D}\,\bot\mathcal{E}$$

- Commutation of $\vee\mathcal{E}$.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E} \qquad \vdots}{D}\,r \qquad \text{to} \qquad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \vdots}{D}\,r \qquad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix} \qquad \vdots}{D}}{D}\,\vee\mathcal{E}$$

- Commutation of $\exists\mathcal{E}$.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists\xi.A\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C}\,\exists\mathcal{E} \qquad \vdots}{D}\,r \qquad \text{converts to} \qquad \dfrac{\begin{matrix}\vdots\\ \exists\xi.A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \vdots}{D}\,r}{C}\,\exists\mathcal{E}$$

Example

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array}}{C\vee D}\,\vee\mathcal{E} \qquad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\,\vee\mathcal{E}$$

converts to

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \qquad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\,\vee\mathcal{E} \qquad \dfrac{\begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array} \qquad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \qquad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\,\vee\mathcal{E}}{E}\,\vee\mathcal{E}$$

Properties of Conversion

- Church-Rosser property still holds.
- The strong normalisation theorem also holds.
- The extension to the full fragment however is very technical.
  - Just count how many rules and conversions we have!
- We will not give the details here.

Curry-Howard Isomorphism

- The full fragment has its corresponding calculus.
  - For $\perp$ and $\vee$, we will add a new type and terms to represent deductions.
  - Observe that conversion rules for terms are derived from conversions of deductions.
- For $\perp$, let Emp be the *empty* type and $\epsilon_U : \mathsf{Emp} \to U$.

$$
\begin{array}{rcll}
\pi^1(\epsilon_{U\times V}t) & \rightsquigarrow & \epsilon_U t & \\
\pi^2(\epsilon_{U\times V}t) & \rightsquigarrow & \epsilon_V t & \\
(\epsilon_{U\to V}t)u & \rightsquigarrow & \epsilon_V t & \\
\epsilon_U(\epsilon_{\mathsf{Emp}}t) & \rightsquigarrow & \epsilon_U t & \\
\delta x.u\ y.v\ (\epsilon_{R+S}t) & \rightsquigarrow & \epsilon_U t & (u, v : U)
\end{array}
$$

Curry-Howard Isomorphism

- For example, consider the following conversion in deductions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \bot\end{array}}{U \Rightarrow V}\,\bot\mathcal{E} \qquad U}{V} \Rightarrow \mathcal{E} \quad \text{converts to} \quad \dfrac{\begin{array}{c}\vdots \\ \bot\end{array}}{V}\,\bot\mathcal{E}$$

- Its corresponding conversion is

$$(\epsilon_{U \to V} t)u \rightsquigarrow \epsilon_V t$$

Curry-Howard Isomorphism

- For $U \vee V$, let $U + V$ be the *sum* type, $\iota^1 : U \to U + V$ and $\iota^2 : V \to U + V$. If $x : R$ and $y : S$ are variables, and $u : U, v : V, t : R + S$ are terms, then $\delta x.u\ y.v\ t : U$. The standard conversions are

$$\delta x.u\ y.v\ (\iota^1 r) \quad \leadsto \quad u[r/x] \qquad\qquad \delta x.u\ y.v\ (\iota^2 s) \quad \leadsto \quad v[s/x]$$

The commuting conversions are

$$\begin{array}{rcll}
\pi^1(\delta x.u\ y.v\ t) & \leadsto & \delta x.(\pi^1 u)\ y.(\pi^1 v)\ t & U = V \times W \\
\pi^2(\delta x.u\ y.v\ t) & \leadsto & \delta x.(\pi^2 u)\ y.(\pi^2 v)\ t & U = V \times W \\
(\delta x.u\ y.v\ t)w & \leadsto & \delta x.(uw)\ y.(vw)\ t & U = V \to W \\
\epsilon_W(\delta x.u\ y.v\ t) & \leadsto & (\delta x.(\epsilon_W u)\ y.(\epsilon_W v)\ t) & U = \mathsf{Emp} \\
\delta x'.u'\ y'.v'\ (\delta x.u\ y.v\ t) & \leadsto & \delta x.(\delta x'.u'\ y'.v'\ u)\ y.(\delta x'.u'\ y'.v'\ v)\ t & \\
 & & & U = V + W
\end{array}$$

Curry-Howard Isomorphism

- Consider the following conversion in deductions:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ R\vee S\end{array}\quad\begin{array}{c}[R]\\ \vdots\\ \bot\end{array}\quad\begin{array}{c}[S]\\ \vdots\\ \bot\end{array}}{\bot}\,\vee\mathcal{E}}{W}\,\bot\mathcal{E} \qquad \text{converts to} \qquad \cfrac{\begin{array}{c}\vdots\\ R\vee S\end{array}\quad \cfrac{\begin{array}{c}[R]\\ \vdots\\ \bot\end{array}}{W}\,\bot\mathcal{E}\quad \cfrac{\begin{array}{c}[S]\\ \vdots\\ \bot\end{array}}{W}\,\bot\mathcal{E}}{W}\,\vee\mathcal{E}$$

- Its corresponding conversion is

$$\epsilon_W(\delta x.u\; y.v\; t) \rightsquigarrow (\delta x.(\epsilon_W u)\; y.(\epsilon_W v)\; t)$$